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Solvable quantum mechanical potentials and their symmetries

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Contents

1 Introduction

Local potentials have been used to model the interactions of the subatomic world ever since the introduction of quantum mechanics. Some of these (like the Coulomb potential) do not differ essentially from the forces observed in nature, while most of them (like the harmonic oscillator, for example) represent approximations of the actual physical situation. The potential shape, defined by the potential type and the parameters in it is usually chosen in a way that reflects the physical picture our intuition associates with the problem; therefore we can define attractive or repulsive, short-range or long-range potentials, etc. The concept of potentials is deeply rooted in the thinking of most physicist. This is perhaps not surprizing, because the most elementary examples introduced at the dawn of quantum mechanics still form essential part of any quantum mechanical course, and also play a fundamental role in the formulation of most physical models of the microscopic world.

Some of the potentials used in quantum mechanics are exactly solvable. This means that the energy eigenvalues, the bound-state wavefunctions and the scattering matrix can be determined in closed analytical form. The range of these potentials has been extended considerably in the recent years by investigations inspired by some novel symmetry-based approaches. The concept of solvability has also been extended: one can talk about conditionally exactly or quasi-exactly solvable potentials too, in addition to the "classical" exactly solvable examples. Due to these developments more and more interactions can be modelled by making advantage of the increasingly flexible potential shapes offered by solvable potentials. Their solutions can be applied directly, or can be combined with numerical calculations. In the simplest case analytical calculations can aid numerical studies in areas where numerical techniques might not be safely controlled. This is the case, for example, when bound-state wavefunctions with arbitrary node numbers are required, for certain singular potentials, or for complex potentials. As the next level of complexity, analytical solutions can supply a basis for numerical calculations. This makes exactly solvable problems indispensable even in the age of rapidly developing computational resources.

Besides their role in describing realistic physical problems, solvable quantum mechanical potentials also represent an interesting field of investigation in their own right. This is largely due to the mathematical elegance and beauty associated with the symmetries of these problems. Symmetries and invariance properties are among the most characteristic features of any physical system. They usually give a deeper insight into the physical nature of the problem,

but also help their mathematical formulation. Symmetries typically lead to characteristic patterns in the energy spectrum of the system. These features are shared by the "classic" potential problems of non-relativistic quantum mechanics. Technically these are relatively simple systems, and accordingly they include a number of exactly solvable examples, nevertheless, they represent the showcase of a wide variety of symmetry and invariance concepts. The most widely known symmetries of quantum mechanical potentials are based on group theory (in particular, Lie algebras), supersymmetry and $\mathcal{PT}$ symmetry.

In group theoretical approaches [1, 2] the elements of various algebras connect different eigenstates of the same Hamiltonian or some interrelated Hamiltonians, while the states themselves belong to the irreducible representations of the corresponding group. A less immediate application of the concept of symmetry appears in supersymmetric quantum mechanics (SUSYQM) [3, 4], where supersymmetry relates two Hamiltonians which typically have (essentially) identical spectra. The most recent symmetry concept is the so called $\mathcal{PT}$ symmetry of one-dimensional quantum mechanical potentials [5], which surprized the theoretical community with non-Hermitian problems possessing real energy spectra.

My purpose with the present work was to investigate solvable potentials from the viewpoint of various symmetry concepts and to explore how these symmetries are related to each other. My intention was to discuss the properties of quantum mechanical problems in the most general framework whenever it was possible. By this I mean that I tried to start with postulating some general construction (an ansätz for the solutions, the differential realization of operators, etc.) and to derive know results as special cases, hoping also that the general procedure leads also to new results.

Throughout this theoretical work, I was also aware of the importance of solvable problems in describing realistic physical problems, and I proposed the utilization of some of my results in this field [P3, 6, P5, 7]. Since my scientific background is in theoretical nuclear physics, I picked most of the prospective applications from this field. My activity in the field of algebraic models of nuclear structure (from where half of my publications originate) prompted me to use my results in describing various cluster–cluster interactions [6] or in developing a high-precision Green-operator method to determine bound and resonance states of the $\alpha - \alpha$ system [7]. However, I did not include these works among the thesis points of my dissertation, because I wanted to keep the homogeneity of the latter as a purely theoretical work. (This also means that occasionally my results were beyond the scope of theoretical physics, as I

found some previously unknown mathematical relations [P16].)

My background in theoretical nuclear physics also determined the methods I was using, i.e. group theory and supersymmetry. My experience in describing complex nuclear physical systems in terms of these symmetry-based methods was of considerable help in discussing various symmetry aspects of quantum mechanical potentials too.

The structure of the dissertation is the following. In section 2 a general introduction is given to solvable potentials of quantum mechanics, with special emphasis on symmetry-based approaches. Besides citing the essential facts from the literature, this section also contains some of my earlier results not included in the theses of the present dissertation. Section 3 contains my main results. These are arranged into five parts: in subsection 3.1 non-trivial examples are presented for the solution of the one-dimensional Schrödinger equation and related problems, subsections 3.2, 3.3 and 3.4 contain my results concerning supersymmetric techniques, Lie-algebraic methods and $\mathcal{PT}$ symmetry in the description of solvable potentials, while in subsection 3.5 the interrelation of these symmetry concepts is discussed. Finally, the summary in section 4 lists the main results of the dissertation according to the thesis points.

In an effort to help the reader I separated typographically those parts, which are less important and can be skipped at the first reading. I also started most subsections by first stating the purpose of the work presented there and pointing out the key motifs.

2 An overview of solvable potentials

This section contains the essential background information necessary for the presentation of my results in section 3. The works cited here include some of my earlier results, which do not appear among my thesis points.

2.1 General aspects of solvable potentials

Various strategies of solving the Schrödinger equation can be used, depending on the nature of the potential. The solutions of the most well–known potentials can be obtained by transforming the Schrödinger equation into the differential equation of some special functions of mathematical physics. This is the case for the Natanzon class of potentials [8], for example, the solutions of which are related to hypergeometric (or confluent hypergeometric) functions. In some other cases the wavefunctions cannot be written in terms of such special functions, nevertheless the techniques described below can still be applied to them.

The procedure presented here was first used to derive some simple potentials [9], but later it was developed further by Natanzon who applied it systematically to transform the Schrödinger equation into the differential equation of the hypergeometric and confluent hypergeometric functions [8]. Following the discussion of [10, 11], let us consider transformation of the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \tag{1}$$

into the second-order differential equation of a special function $F(z)$. For this, we search for the solutions in the form

$$\psi(x) = f(x)F(z(x)) \ . \tag{2}$$

At the moment we do not specify the domain of definition for the coordinate x itself. Later on we shall come back to this issue and its importance for $\mathcal{PT}$ symmetric problems.

Once we substitute (2) in the Schrödinger equation (1), we arrive at the ordinary differential equation of the special function $F(z)$

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + Q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + R(z)F(z) = 0 \tag{3}$$

where, by construction

$$\frac{z''}{(z')^2} + \frac{2f'}{z'f} = Q(z(x)) \tag{4}$$

and

$$\frac{f''}{(z')^2 f} + \frac{E - V(x)}{(z')^2} = R(z(x)) \ . \tag{5}$$

From these equations an explicit expression follows for $E - V(x)$:

$$E - V(x) = (z'(x))^2 R(z(x)) - \left(\left(\frac{f'(x)}{f(x)} \right)^2 + \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f'(x)}{f(x)} \right) \right) \tag{6}$$

$$= \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + (z'(x))^2 \left(R(z(x)) - \frac{1}{2} \frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4} Q^2(z(x)) \right) \tag{7}$$

Besides the functions $Q(z)$ and $R(z)$ defining the special function $F(z)$, (7) contains only the function representing a variable transformation, $z(x)$. This also applies to the solutions themselves:

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp \left(\frac{1}{2} \int^{z(x)} Q(z) \mathrm{d}z \right) F(z(x)) \ . \tag{8}$$

We are left with the task of finding a functional form of $z(x)$ which transforms the Schrödinger equation (7) into an exactly solvable problem.

Of course, any randomly chosen $z(x)$ function satisfies the latter ambitious requirement for a particular potential $V(x)$ and energy E. We only cannot guarantee in general that any other physical solution of the same physical problem can be found in the same manner as well. In this perspective, a useful way of finding reasonable $z(x)$ functions has been proposed by Bhattacharjie and Sudarshan [9]. According to them, if there is a constant (E) on the left-hand side of (7), then there must be one on the right-hand side too. In [10] this fact was exploited, and a systematic list of potentials was compiled by identifying certain terms found on the right-hand side of (7) with a constant C. This assignment leads to first-order differential equations for z of the type

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x} \right)^2 \phi(z) = C \ , \tag{9}$$

where $\phi(z)$ is a function of z originating from $R(z)$ or $Q(z)$.

The general solution of the latter differential equation is given by formula

$$\int \phi^{1/2}(z) \mathrm{d}z = C^{1/2} x + \delta \ . \tag{10}$$

This defines an implicit function $x(z)$ and, in many cases of practical interest, also the explicit $z(x)$ function we need [10]. Usually $\delta = 0$ is considered in order

to set $z(0) = 0$. The $\delta \neq 0$ choice corresponds to a shift of the coordinate and reflects a trivial and also rarely relevant transformation for potentials defined on the real x axis, but we shall find it important in connection with $\mathcal{PT}$ symmetric potentials discussed later on in subections 2.4 and 3.4.

The general Natanzon class potentials can be obtained from a systematic application of this transformation procedure to the hypergeometric and confluent hypergeometric functions. (In the latter case potentials are sometimes called Natanzon confluent potentials [12].) Starting from hypergeometric functions the most general potential can be written as [8]

$$V(z(x)) = -\frac{1}{2}\frac{z'''}{z'} + \frac{3}{4}\left(\frac{z''}{z'}\right)^2 + \frac{fz(z-1) + h_0(1-z) + h_1 z}{R(z)}, \qquad (11)$$

where

$$R(z) = a_1 z(z-1) + c_1 z + c_0(1-z). \qquad (12)$$

$z(x)$ is then determined from the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2z(1-z)}{R^{1/2}(z)}, \qquad (13)$$

which is determined from the current version of (9) after identifying the linear combination of three independent terms on its right-hand side with a constant. The three coefficients (a_1, c_1 and c_0) govern the behaviour of $z(x)$ and supply three of the six potential parameters. (The other three parameters are f, h_0 and h_1 in (11).) In general, there is no explicit expression for the energy eigenvalues, rather they can be determined from the implicit formula

$$\begin{aligned} 2n+1 &= (f+1-a_1E_n)^{1/2} - (h_0+1-c_0E_n)^{1/2} - (h_1+1-c_1E_n)^{1/2} \\ &\equiv \alpha_n - \beta_n - \delta_n, \end{aligned} \qquad (14)$$

while the wavefunctions are written as

$$\psi_n(x) \simeq R^{1/4}(z(x))(z(x))^{\beta_n/2}(1-z(x))^{\delta_n/2}\ {}_2F_1(-n, \alpha_n - n; \beta_n + 1; z(x)). \qquad (15)$$

Natanzon confluent [12] potentials

$$V(h(x)) = -\frac{1}{2}\frac{h'''}{h'} + \frac{3}{4}\left(\frac{h''}{h'}\right)^2 + \frac{g_1h^2 + g_1h + \eta}{R(h)}, \qquad (16)$$

with $R(h) = \sigma_2 h^2 + \sigma_1 h + c_0$ can be obtained from (11) by the substitutions $a_1 = \sigma_2/\tau^2$, $c_1 - a_1 - c_0 = \sigma_1/\tau$, $h_1 - h_0 - f = g_1/\tau$, $z = h\tau$ and taking the limit $\tau \to 0$.

Although equations (11) to (15) contain all the necessary formulae to get the solutions of any Natanzon class potential, calculations become rather involved in general. In many cases, for example, it is impossible to get $z(x)$, the solution of (13) in closed form, rather only an implicit $x(z)$ function can be determined. This is the case for the so-called "implicit" potentials, like the Ginocchio [13], the generalized Ginocchio [14], the PIII [15] or the (Natanzon confluent type) generalized Coulomb potential [16]. Once knowing $x(z)$, $z(x)$ can, of course, be determined to any desired accuracy. Similarly, E_n is also contained implicitly in (14) in general, as is the case for the potential in [17].

These complications hinder the application of these formulae, except for some special subcases, which correspond to the most well-know textbook examples for solvable potentials. These are the so-called shape-invariant potentials [18], the definition of which will be given later in subsection 2.2, in connection with supersymmetric quantum mechanics. However, considering them in the context of the Natanzon potential class, there is a simple rule that identifies them. In particular, they correspond to cases in which only one of the parameters a_1, c_0 and c_1 (in (12)) is non-zero, and this results a straightforward solution for both $z(x)$ and E_n from equations (13) and (14), respectively. Table 1 lists the known shape-invariant potentials following the presentation of [10], which was the first complete (as it is known presently) compilation of these potentials, and where a natural classification scheme was also proposed for them. In [10] shape-invariant potentials have been obtained from a systematic application of the method described previously to orthogonal polynomials. Substituting the Jacobi, generalized Laguerre and Hermite polynomials [19] in (2) as $F(z)$ and identifying certain terms on the right-hand side of (7) with the constant (energy) term has led to differential equations of the type (9) which defined a straightforward classification scheme of these potentials. The resulting potentials are displayed in table 1, along with the differential equations defining the individual classes [10].

> It has to be noted that there is some redundancy in the shape-invariant potentials listed in table 1. The generalized Pöschl–Teller and the Pöschl–Teller II potentials are essentially identical, as are their trigonometric versions, the Pöschl–Teller I and the Sacrf I potentials, as can be seen from the corresponding $z(x)$ functions, which differ only in an $x \leftrightarrow 2x$ scale transformation. Also, the radial and the one-dimensonal harmonic oscillators differ from each other in the boundary conditions. These redundant potentials are still often mentioned as separate examples due to historical reasons.

The "implicit" and other potentials mentioned earlier occupy an interme-

Table 1: The 12 shape-invariant potentials and their interpretation as Natanzon-class potentials. The classes denoted with P, L and H correspond to potentials containing the Jacobi, generalized Laguerre and Hermite polynomials in their bound-state solutions.

Diff. eq. (Class) $z(x)$	$V(x)$	Name
$\frac{(z')^2}{(1-z^2)} = C$ (PI)		
$\mathrm{i}\sinh(ax)$, $C = -a^2$	$A^2 + (B^2 - A^2 - Aa)\mathrm{sech}^2(ax)$ $B(2A + a)\mathrm{sech}(ax)\tanh(ax)$	Scarf II
$\cosh(ax)$, $C = -a^2$	$A^2 + (B^2 + A^2 + Aa)\mathrm{cosech}^2(ax)$ $-B(2A + a)\mathrm{cosech}(ax)\coth(ax)$	generalized Pöschl–Teller
$\cosh(2ax)$, $C = -4a^2$	$(A - B)^2 - A(A + a)\mathrm{sech}^2(ax)$ $+B(B - a)\mathrm{cosech}^2(ax)$	Pöschl–Teller II
$\cos(ax)$, $C = a^2$	$-(A + B)^2 + A(A - a)\sec^2(ax)$ $+B(B - a)\mathrm{cosec}^2(ax)$	Scarf I
$\cos(2ax)$, $C = 4a^2$	$-(A + B)^2 + A(A - a)\mathrm{sec}^2(ax)$ $+B(B - a)\mathrm{cosec}^2(ax)$	Pöschl–Teller I
$\frac{(z')^2}{(1-z^2)^2} = C$ (PII)		
$\tanh(ax)$, $C = a^2$	$A^2 + B^2/A^2 + 2B\tanh(ax)$ $-A(A + a)\mathrm{sech}^2(ax)$	Rosen–Morse II
$\coth(ax)$, $C = a^2$	$A^2 + B^2/A^2 - 2B\coth(ax)$ $+A(A - a)\mathrm{cosech}^2(ax)$	Eckart
$-\mathrm{i}\cot(ax)$, $C = -a^2$	$-A^2 + B^2/A^2 - 2B\cot(ax)$ $+A(A + a)\mathrm{cosec}^2(ax)$	Rosen–Morse I
$(z')^2/z = C$ (LI)		
$\frac{\omega}{2}x^2$, $C = 2\omega$	$\frac{1}{4}\omega^2x^2 + \frac{l(l+1)}{x^2} - (l + \frac{3}{2})\omega$	radial HO
$(z')^2 = C$ (LII)		
$\frac{\mathrm{e}^2}{n+l+1}x$, $C = \frac{\mathrm{e}^4}{(n+l+1)^2}$	$\frac{e^4}{4(l+1)^2} - \frac{e^2}{x} + \frac{l(l+1)}{x^2}$	Coulomb
$(z')^2/z^2 = C$ (LIII)	$A^2 - B(2A + a)\exp(-ax)$	Morse
$\exp(-ax)$, $C = a^2$	$+B^2\exp(-2ax)$	
$(z')^2 = C$ (HI)		
$(\frac{\omega}{2})^{1/2}(x - \frac{2b}{a})$, $C = \omega/2$	$-\frac{1}{2}\omega + \frac{1}{4}\omega^2x^2$	one-dim. HO

diate situation between the general Natanzon (confluent) potentials and their shape-invariant subclass. In particular, for potentials in [13, 15, 16] two of the three parameters determining the variable transformation $z(x)$ (in (12) and (13)) are non-zero, which simplifies the determination of E_n from (14), but results an implicit $x(z)$ function. The potentials in [17] represent another kind of special case, in some sense opposite to the previous examples, as a special arrangement of these three parameters ($a = 4c_1 = 4c_0$) leads to an explicit $z(x)$ function, but at the same time results a complicated implicit energy formula. In principle, the Woods–Saxon potential [20] has similar characteristics, but due to the boundary conditions its energy eigenvalues have to be determined from a transcendent equation.

While still manageable algebraically, these Natanzon-class potentials offer potential shapes different from the simplest solvable problems. For example, the Ginocchio [13, 14] and the generalized Coulomb [16] potential are similar to some nuclear physical and screened Coulomb potentials, respectively, while the PIII potential of [15] corresponds to some typical molecular physical potential with a "pocket". This flexibility of shape also means that the Ginocchio [13, 14] and the generalized Coulomb [16] potentials have shape-invariant limits.

The simple transformation procedure outlined previously can be applied to any other function satisfying second-order differential equations of the form (3). It is possible to transform the differential equation of the Bessel functions into the Schrödinger equation with the potential $V(r) \simeq \exp(-r/a)$ (with $l = 0$) and that of a particle enclosed in a sphere [20], however, the Bessel functions are less appropriate for this treatment than the orthogonal polynomials. One reason is that the $R(z)$ function in (3) and (7) has more complex structure, $R(z) = 1 - \nu^2/z^2$, and this indirectly means that the wavefunctions are more difficult to handle, furthermore, the energy eigenvalues cannot be written in a closed form, rather they have to be determined from the zeros of the Bessel functions.

In some cases the Schrödinger equation cannot be solved by the transformation procedure used previously, as the wavefunctions cannot be written in terms of known special functions. An example for this situation is the family of quasi-exactly solvable (QES) potentials [21]. These potential problems cannot be solved exactly in general, except for some special values of the parameters, when a finite number of exact energy eigenvalues and eigenfunctions can be determined. Typical examples for QES potentials are anharmonic oscillators and polynomial-type potentials. In this case the basis states are constructed as polynomials times an exponential factor containing also polynomials. Substituting them into the Schrödinger equation and then matching the terms

with various powers typically leads to recursion relations for the coefficients appearing in the basis states. The recursion series can then be terminated by an appropriate choice of the parameters, which means that the first few basis states can be obtained in closed form.

The most recent concept of solvability is related to conditionally exactly solvable (CES) potentials. The first models coined CES potentials [22, 23] were characterized by the fact that the coupling constant of some potential term had to be fixed to a numerical constant value in order to obtain their solutions. Some of these potentials [24] can be obtained by the ad hoc modification of the Natanzon confluent potentials, e.g. by changing the powers of h in (16). The techniques of supersymmetric quantum mechanics (to be reviewed in subsection 2.2) offer further ways to generate exactly solvable potentials from known ones, some of which have been classified as CES potentials [25].

2.2 Supersymmetric quantum mechanics

Supersymmetric theories describe bosons and fermions in a unified way, therefore their algebraic formulation makes use of commutators as well as anticommutators. These theories appeared first in quantum field theoretical studies, where they turned out to be less divergent than conventional theories. However, if supersymmetry exists in nature, then it must be broken, because there is no evidence for degenerate boson–fermion multiplets, and this fact has generated interest in supersymmetry breaking mechanisms. This is how supersymmetric quantum mechanics (SUSYQM) was born [26], but it soon began to live its own life.

In quantum mechanics the most widely used construction is $N = 2$ supersymmetry, in which the supersymmetric Hamiltonian $\mathcal{H}$ and the Q and $Q^\dagger$ supersymmetric charge operators satisfy the following relations [3, 4]

$$\{Q, Q^\dagger\} = \mathcal{H} \qquad\qquad Q^2 = (Q^\dagger)^2 = 0$$

$$[Q, \mathcal{H}] = [Q^\dagger, \mathcal{H}] = 0, \tag{17}$$

i.e. the charge operators are nilpotent and commute with the supersymmetric Hamiltonian.

The realization of this superalgebra is usually given in terms of 2×2 matrices:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad\qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \tag{18}$$

where A and $A^\dagger$ are some operators to be specified later. These matrices together with

$$\mathcal{H} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{19}$$

represent a realization of the algebra (17), which is also recognized as the $sl(1/1)$ Lie superalgebra. Based on the above matrix realization of these operators we can interpret the supersymmetric Hamiltonian $\mathcal{H}$ as a composition of two scalar Hamiltonians H_- and H_+, which act in the "bosonic" and the "fermionic" sector of the two–component basis states. These two sectors are connected by the charge operators as

$$Q \begin{pmatrix} \psi^{(-)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi^{(-)} \end{pmatrix}, \qquad Q^\dagger \begin{pmatrix} 0 \\ \psi^{(+)} \end{pmatrix} = \begin{pmatrix} A^\dagger\psi^{(+)} \\ 0 \end{pmatrix}, \tag{20}$$

where $\psi^{(-)}$ and $\psi^{(+)}$ stand for eigenstates of the "bosonic" Hamiltonian $H_- = A^\dagger A$ and the "fermionic" Hamiltonian $H_+ = AA^\dagger$. H_- and H_+ are called supersymmetric partners.

The existence of the superalgebra (17), and in particular, that of the supersymmetric charges commuting with the supersymmetric Hamiltonian has important implications regarding the energy spectra of H_- and H_+ [3, 4]. First of all, it follows from their construction that the eigenvalues of $H_- = A^\dagger A$ and $H_+ = AA^\dagger$ are non–negative. Let us assume that $\psi^{(-)}$ and $\psi^{(+)}$ are normalized eigenfunctions of H_- and H_+, with eigenvalues $E^{(-)}$ and $E^{(+)}$, respectively:

$$H_-\psi^{(-)} = E^{(-)}\psi^{(-)} \qquad H_+\psi^{(+)} = E^{(+)}\psi^{(+)}. \tag{21}$$

The simple equation

$$H_+(A\psi^{(-)}) = AA^\dagger(A\psi^{(-)}) = AH_-\psi^{(-)} = E^{(-)}A\psi^{(-)} \tag{22}$$

clearly shows that $E^{(-)}$ is also an energy eigenvalue of H_+, and the corresponding normalized eigenfunction is

$$\psi^{(+)} = (E^{(-)})^{-1/2}A\psi^{(-)}, \tag{23}$$

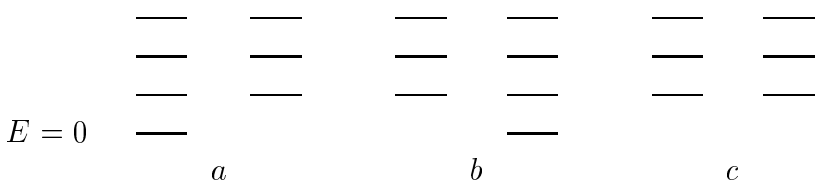
except when $A\psi^{(-)} = 0$ holds. Similar arguments apply to the reverse situation when the roles of H_- and H_+ are exchanged:

$$H_-(A^\dagger\psi^{(+)}) = A^\dagger A(A^\dagger\psi^{(+)}) = A^\dagger H_+\psi^{(+)} = E^{(+)}A^\dagger\psi^{(+)}, \tag{24}$$

i.e. $E^{(+)}$ is an allowed eigenvalue of H_- too, with the normalized eigenfunction

$$\psi^{(+)} = (E^{(+)})^{-1/2}A^\dagger\psi^{(+)}, \tag{25}$$

Figure 1: Schematic sketch of the possible arrangement of the energy spectra of the supersymmetric partner potentials H_- and H_+.



except when $A^\dagger\psi^{(+)} = 0$ holds.

This kind of relationship between the supersymmetric partner Hamiltonians leads to three possible patterns of their energy spectra. (See figure 1).

i) Whenever $A\psi_0^{(-)} = 0$ holds for a normalizable eigenstate of H_-, $H_-\psi_0^{(-)} = A^\dagger A\psi_0^{(-)} = 0$ implies that this eigenstate is also the ground state of H_- with $E_0^{(-)} = 0$ eigenenergy. This argument holds in the reverse direction too, as $0 = E_0^{(-)} = \langle\psi_0^{(-)}|A^\dagger A|\psi_0^{(-)}\rangle = |A|\psi_0^{(-)}\rangle|^2$ leads to $A\psi_0^{(-)} = 0$. In this case H_+ has no normalizable eigenstate with zero energy, and we get the situation depicted in panel a of figure 1 with

$$E_{n+1}^{(-)} = E_n^{(+)}, \qquad n = 0,\ 1,\ 2,\ ..., \qquad E_0^{(-)} = 0. \tag{26}$$

The eigenstates of H_- and H_+ lying at the same energy are related to each other as $\psi_n^{(+)} = (E_{n+1}^{(-)})^{-1/2}A\psi_{n+1}^{(-)}$ and $\psi_{n+1}^{(-)} = (E_n^{(+)})^{-1/2}A^\dagger\psi_n^{(+)}$.

ii) The same line of reasoning applies to the case in which the roles of H_- and H_+ are interchanged and the energy spectra are similar to those shown in the b panel of figure 1.

iii) If neither H_- nor H_+ have normalizable eigenstate with zero energy, the spectra of H_- and H_+ turn out to be identical, as shown in the c panel of figure 1. This case is known to correspond to broken supersymmetry.

Until this point the operators were considered as abstract mathematical quantities which satisfy some prescribed relations, and were not specified in more detail. Now we shall consider a specific differential realization of A and $A^\dagger$ from which the one–dimensional Schrödinger equation (with $\hbar = 2m = 1$) can be recovered:

$$H_\pm\psi^{(\pm)}(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x)\right)\psi^{(\pm)}(x) = E^{(\pm)}\psi^{(\pm)}(x). \tag{27}$$

One possible choice for this is

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \qquad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x), \tag{28}$$

which recovers (27) with the potentials

$$V_\pm(x) = W^2(x) \pm \frac{\mathrm{d}}{\mathrm{d}x} W(x). \tag{29}$$

$W(x)$, the superpotential is uniquely related to the ground–state wavefunction of H_- via $A\psi_0^{(-)} = 0$:

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^{(-)}(x), \tag{30}$$

or

$$\psi_0^{(-)}(x) = N_0 \exp\left(-\int^x W(y)\mathrm{d}y\right), \tag{31}$$

where N_0 is a normalization constant.

An immediate consequence of these results is that whenever the solutions of a one-dimensional potential $V_-(x)$ are known, the solutions of its supersymmetric partner potential

$$V_+(x) = V_-(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \psi_0^{(-)}(x) \tag{32}$$

can also be obtained, furthermore the bound-state energy spectrum of the two potentials are related by (26). This procedure can be followed for any potential (after possibly a simple shift of the energy scale setting $E_0^{(-)} = 0$), furthermore it can be applied to potentials solved by either analytical or numerical methods.

A further remarkable aspect of SUSYQM is that a whole series of isospectral solvable potentials can be constructed by consecutive application of this procedure. Adjacent members of this hierarchy of potentials are supersymmetric partners, and each potential has one less bound state than the one constructed at the previous stage.

Here we may note that (6) offers a straightforward connection to the formalism of SUSYQM. In particular, whenever $R(z)$ vanishes for the ground state, we have

$$E - V(x) = -W^2(x) + \frac{\mathrm{d}W}{\mathrm{d}x}, \tag{33}$$

with the superpotential being

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x) \tag{34}$$

$$= -\frac{1}{2} Q(z(x)) z'(x) + \frac{1}{2} \frac{z''(x)}{z'(x)}. \tag{35}$$

The condition $R_{n=0}(z) = 0$ always holds for the orthogonal polynomials, making them an ideal subject of the SUSYQM approach.

Despite their similar bound-state energy spectra, supersymmetric partner potentials constructed by (32) usually have different structure (both in the geometric and algebraic sense). In some cases, however, $V_-(x)$ and $V_+(x)$ have the same functional dependence on the coordinate and differ only in some potential parameters which set their depth and shape. This is the case with the shape-invariant potentials [18] mentioned previously in subsection 2.1. These potentials are defined by the relationship

$$V_+(x;a_0) - V_-(x;a_1) \equiv W^2(x;a_0) + W'(x;a_0) - W^2(x;a_1) + W'(x;a_1) = R(a_1), \tag{36}$$

where a_0 and a_1 stand for parameters of the supersymmetric partner potentials, and $R(a)$ is a constant. The two sets of potential parameters a_0 and a_1 are connected by simple mathematical formulae written formally as

$$a_1 = f(a_0). \tag{37}$$

This f function turned out to be a simple addition: $a_{i+1} = a_i + const.$ for the shape-invariant potentials [18]. Equation (36) shows that for shape-invariant potentials the consecutive application of a SUSY transformation (32) and a change of the potential parameters as in (37) recovers the original potential, apart from an energy shift. It can be shown [18] that the discrete energy spectrum of $V_-(x;a_0)$ can be written as

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k), \tag{38}$$

where a_k is generated by the consecutive application of f in (37):

$$a_k = f^k(a_0). \tag{39}$$

Besides their energy eigenvalues, the wavefunctions of these shape-invariant potentials can also be computed in a straightforward way. Combining the fact that these potentials have the same functional form and that a whole hierarchy of potentials can be generated in terms of SUSYQM, it was shown [27] that the wavefunctions $\psi_n^{(-)}(x;a_0)$ of potential $V_-(x;a_0)$ can be obtained by consecutive application of the SUSYQM laddering operators $A^\dagger(x;a_k)$, as

$$\psi_n^{(-)}(x;a_0) = N_0 A^\dagger(x;a_0)A^\dagger(x;a_1).....A^\dagger(x;a_{n-1})\psi_0^{(-)}(x;a_n). \tag{40}$$

As it has been mentioned already in subsection 2.1, shape-invariant potentials turned out to play a distinguished role among solvable potentials. This is mainly due to their feature that acting on their bound-state wavefunctions with the linear differential operators A, the resulting function can be rewritten in terms of a *single* wavefunction of the type (2), i.e. it is recovered in essentially the same form as the original wavefunction, except for the potential parameters appearing in it. This is not the case for the general Natanzon-class [8] potentials: in that case the resulting wavefunction contains two terms with two *separate* special function of the type $F(z)$, i.e. the partner potential $V_+(x)$ is *outside* the Natanzon class. This peculiar feature of shape-invariant potentials prompted a search for potentials with this property. Several attempts have been made to identify and classify all shape-invariant potentials [28, 10, 29] and the results suggest that finding such potentials in addition to the known ones listed in table 1 is unlikely. It was also shown [30] that several of these 12 potentials have been "rediscovered" in one way or another. In addition to the bound-state spectra and the wavefunctions of these potentials scattering amplitudes have also been calculated [31] for them.

It is notable that the classification scheme followed in table 1 [10] is basically the same as that of Infeld and Hull [32] based on the factorization method [33] and that of Miller [34] originating from the Lie theory of special functions, while it differs from the classification scheme of Cooper et al. [28] which was derived following the ideas of SUSYQM.

We note that there were attempts to generalize the concept of shape-invariance to systems where the f function in (39) is a multiplicative, rather than an additive function [35], but this hasn't resulted in meaningful potentials.

As it has been mentioned previously, eliminating the ground state of a potential is only one of the possible transformations handled by SUSYQM (see figure 1). The remaining cases, i.e. altering the potential while inserting a new ground state or leaving the spectrum unchanged can be implemented in a similar way, except that instead of the ground-state wavefunction, some unphysical (but nodeless) solutions have to be used. The boundary conditions of these solutions will then determine the effect of the SUSY transformation on the energy spectrum and on the potential [36, 37].

Here we consider the radial Schrödinger equation

$$H\psi(r) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_0(r)\right)\psi(r) = E\psi(r), \tag{41}$$

but a similar treatment of the one-dimensional motion ($-\infty < x < \infty$) can also be formulated by imposing different boundary conditions on the physical solutions of (41) [38]. In what follows we assume that $V_0(r)$ already contains the centrifugal term $l(l+1)r^{-2}$ with an orbital angular momentum l which remains fixed, and that $V(r)$ behaves like $V(r) \simeq m(m+1)r^{-2}$ near the origin, where m is a positive integer, i.e. it can have additional singularity if $m \neq l$ holds. We also change the notation somewhat: potentials (Hamiltonians) linked by these transformations may still be considered supersymmetric partners, nevertheless their labelling with + and − may become confusing in some situations, therefore we shall abandon these symbols.

Consider the factorization

$$H = A^\dagger(\epsilon)A(\epsilon) + \epsilon \tag{42}$$

of H in (41), with

$$A(\epsilon) = (A^\dagger(\epsilon))^\dagger = -\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\phi, \tag{43}$$

where ϕ is a solution of $H\phi = \epsilon\phi$. $\phi(r)$ need not be a physical solution in order to generate reasonable potentials in this more general approach, however it has to be nodeless, otherwise the resulting potential would have singularities for $r \neq 0$. Following the procedure presented previously in this section, the SUSY partner of $V_0(r)$ in (41) can be defined as

$$V_1(r) = V_0(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\phi(r), \tag{44}$$

with the difference that $\phi(r)$ may be different from the ground-state wavefunction. If we take $\epsilon = E_0$ and $\phi(r) = \psi_0(r)$, we, of course, arrive at the special case discussed in in detail previously, i.e. the transformation which eliminates the ground-state wavefunction of $V_0(r)$, but leaves the rest of the energy spectrum unchanged.

The nodelessness of $\phi(r)$ can be secured if we consider a factorization energy $\epsilon = -\gamma^2 < E_0$ (with $\gamma > 0$). In contrast with the usual procedure for physical solutions, now we can consider two independent solutions of $H\phi = \epsilon\phi = -\gamma^2\phi$. Taking appropriate linear combinations of the two independent solutions, four types of solutions can be constructed, depending on whether they are regular or irregular at the origin and asymptotically.

These four types of transformations (denoted usually by T_1, T_2, T_3 and T_4 [36]) have characteristic effect on the singularities of the potentials near the

Table 2: SUSYQM transformations belonging to different types of solutions $\phi(r)$ [36, 37]. Here $\epsilon = -\gamma^2 \leq E_0$.

Transformation	T_1	T_2	T_3	T_4
ϵ	$\epsilon = E_0$	$\epsilon < E_0$	$\epsilon < E_0$	$\epsilon < E_0$
$\lim_{r\to 0}\phi$	r^{m+1}	r^{-m}	r^{m+1}	r^{-m}
$\lim_{r\to\infty}\phi$	$\exp(-\gamma r)$	$\exp(\gamma r)$	$\exp(\gamma r)$	$\exp(-\gamma r)$
Spectrum modification	deletes ground state	adds new ground state $(0 < m)$	none	none $(0 < m)$
Singularity modification	$(m+1)r^{-2}$	$-mr^{-2}$	$(m+1)r^{-2}$	$-mr^{-2}$
Phase shift modification	$-\tan^{-1}(k/\gamma_0)$	$-\tan^{-1}(k/\gamma)$	$-\tan^{-1}(k/\gamma)$	$-\tan^{-1}(k/\gamma)$

origin, as well as on the phase shifts of the scattering wavefunctions. These quantities are related to the boundary conditions of the solution $\phi(r)$ of $H\phi = \epsilon\phi$, as simple calculations starting from equations (43) and (44) reveal in each case. The results are summarized in table 2 [37]. Note that transformations T_2 and T_4 work only if $m > 0$ holds, i.e. if the original potential $V_0(r)$ has already had a singularity.

It has to be mentioned here that there is a conceptual difference between Sukumar's [36] and Baye's [37] interpretation of the four types of transformations. Sukumar originally assumed that these transformations change the angular momentum with one unit, which accounts for the difference of the singularity between $V_0(r)$ and $V_1(r)$. However, Baye has pointed out that the value of the orbital angular momentum has already been set when the radial Schrödinger equation (41) was written down [37], therefore the centrifugal term should remain unchanged during the whole procedure.

These results can be applied to the one-dimesional motion too, after modification of the boundary conditions [36], i.e. getting rid of x^{-2}-like singularities.

Finally, it has to be noted that although here we have formulated these transformations in terms of the factorization method which is closer to the formalism of SUSYQM, the whole procedure can also be recognized as the

Darboux transformation [39] of second-order differential equations. A brief review on various approaches to isospectral potentials in terms of the Darboux construction can be found in [40], for example.

An even more sophisticated supersymmetric construction can be obtained by iterating the single SUSY transformations mentioned up to this point. Using pairs of such transformations one can construct [37] potentials that lead to the same phase shifts as the original potential, despite the different number of bound states the two potentials support, and this result was interpreted in terms of the generalized Levinson theorem [41]. This aspect of SUSYQM also allowed straightforward interpretation of the long standing problem represented by the duality of "deep" and "shallow" type potentials used in the description of interacting composite particles. The relation of SUSYQM to other methods of analyzing isospectral potentials, such as the inverse scattering theory [42] has also been discussed [36, 43, 11].

More recently the formalism of generating phase-equivalent potentials has been developed to a stage where, in principle, arbitrary modifications of the energy spectrum are possible [44, 43, 45]. The final potential and the wavefunctions are expressed in terms of compact formulae depending on integrals and determinants composed of physical and unphysical solutions of the Schrödinger equation. These expressions can be evaluated by numerical techniques in general.

Pairs of single SUSY transformations can be employed to generate potentials phase-equivalent with the original one provided that the factorization energies are chosen to be equal, guaranteeing that the original scattering phases are restored after the second step. For the resulting transformation, the factorization energy is not anymore required to be smaller than $E_0^{(0)}$, the ground-state energy of $V_0(r)$. As described in [44], for example, only three non-trivial combinations are possible, and the resulting potential is written as

$$V_2(r) = V_0(r) + 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{(\varphi_0(k_0,r))^2}{\beta + \int_r^\infty (\varphi_0(k_0,t))^2 \mathrm{d}t} \ . \tag{45}$$

The appropriate choices of $\varphi_0(k_0,r)$ and β are summarized in table 3, where the properties of the three basic transformation types are also given. In table 3, $\psi_0^{(i)}$ represents the wavefunction of an arbitrary bound state at energy $E_0^{(i)} = k_0^{(i)2}$ while f_0 represents a solution decreasing at infinity and singular at the origin, at any negative energy $E_0 = k_0^2$ where there is no bound state. The integral in the denominator of (45) always converges because the chosen $\varphi_0(k_0,r)$ decrease exponentially at large r, in all cases. The wavefunctions of $V_2(r)$ are expressed in terms of the original wavefunctions $\varphi_0(k,r)$ and the (physical or unphysical)

Table 3: Properties of the three transformations resulting in potentials phase-equivalent with the original potential $V_0(r)$. We suppose that $V_0(r)$ is singular at the origin as $V_0(r) \simeq m(m+1)r^{-2}$, which accounts for the centrifugal term too.

Transformation	Removes a bound state	Adds a bound state ($m > 1$ only)	Unchanged spectrum
Solution φ_0	$\psi_0^{(i)}$	f_0	$\psi_0^{(i)}$
Parameter β	-1	$\alpha > 0$	$\alpha/(1-\alpha)$, $\alpha > 0$
Fact. energy E_0	$E_0^{(i)} < 0$	$E_0 \neq E_0^{(i)}$, $E_0 < 0$	$E_0^{(i)} < 0$
$\lim_{r\to 0}\varphi_0$	r^{m+1}	r^{-m}	r^{m+1}
$\lim_{r\to\infty}\varphi_0$	$\exp(-\lvert k_0^{(i)}\rvert r)$	$\exp(-\lvert k_0\rvert r)$	$\exp(-\lvert k_0^{(i)}\rvert r)$
Singularity of V_2	$(m+2)(m+3)r^{-2}$	$(m-2)(m-1)r^{-2}$	$m(m+1)r^{-2}$
$F_2(k)/F_0(k)$	$k^2/(k^2+\lvert k_0^{(i)}\rvert^2)$	$(k^2+\lvert k_0\rvert^2)/k^2$	1

factorization functions $\varphi_0(k_0, r)$ as

$$\varphi_2(k,r) = \mathcal{N}^{-\frac{1}{2}}\Big(\varphi_0(k,r) - \varphi_0(k_0,r)\frac{\int_r^\infty \varphi_0(k_0,t)\varphi_0(k,t)\mathrm{d}t}{\beta + \int_r^\infty(\varphi_0(k_0,t))^2\mathrm{d}t}\Big)\ . \qquad (46)$$

Here $\mathcal{N} = 1$, except for $k = k_0$ when $\varphi_2(k_0, r)$ is physical, in which case $\mathcal{N} = \alpha$ [11].

Further potentials phase-equivalent with $V_0(r)$ can be derived by iterating transformation pairs [43, 45]. The equivalent of (45) for multiple spectrum modifications can be written as a compact formula involving a determinant containing integrals of physical and unphysical solutions satisfying (1) [43, 45],

$$V_{2m}(r) = V_0(r) - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\det\left(\beta_i\delta_{ij} + \int_r^\infty \varphi_0(k_i,t)\varphi_0(k_j,t)dt\right), \qquad (47)$$

where the k_i correspond to m different factorization energies $E_i = k_i^2$. A similar formula is available for a generalization of (46) with equations (20) and (21) of [45]. These transformations are, in principle, capable of generating arbitrary modifications of the energy spectrum while keeping the scattering phases unchanged. The determinant form also suggests that the final result is independent of the sequence of the individual transformation pairs, because

exchanging two of them merely corresponds to exchanging two columns of the determinants.

Before closing this subsection, we briefly mention some generalizations of the methods based on supersymmetry and factorization. First we note that the isospectrality of Hamiltonians can be generated by surprisingly simple constructions that do not even refer to the explicit realization of the operators involved. The intertwining relation [47, 48] between Hamiltonians H_1 and H_2

$$H_1 Q = Q H_2 \tag{48}$$

guarantees, for example, that if there exists an eigenstate ψ_2 of H_2 with eigenvalue $E^{(2)}$, then $Q\psi_2$ will be an eigenstate of H_1 with the same eigenvalue. (Note that if Q has an inverse, then the above relation can be interpreted as a similarity transformation between H_1 and H_2.) However, based only on (48) nothing more can be said about the energy spectra in general. A particular realization of (48) can be obtained by assuming that the two Hamiltonians are factorized as

$$H_1 = QR \ , \qquad H_2 = RQ \ . \tag{49}$$

With the additional requirement $Q = R^\dagger$ the Hermiticity of the Hamiltonians and the non-negativity of their eigenvalues can also be guaranteed.

It is remarkable that these rather general results of the factorization method [33, 32] hold in their abstract form, without specifying the realization of the operators. In most cases the Schrödinger equation is factorized in one dimension, i.e. on the $x = (-\infty, \infty)$ or the $[0, \infty)$ intervals (in case of radial equations) or on a finite interval. Then the Q and R operators are naturally chosen as linear differential operators. Combining operators of the type Q and R with matrices gives rise to various SUSYQM constructions, as we have seen previously in the present subsection.

The construction leading to isospectral Hamiltonians can be generalized in several ways. One possibility is considering differential operators of higher order in the realization of the Q and R-type operators [49, 48]. Another possibility is using larger matrixes, as in parasupersymmetric quantum mechanics [50], which replaces the fermionic degrees of freedom with parafermionic ones. This latter theory has also been connected with quantum algebras, and some solvable potentials have been discussed in this context [51].

2.3 Lie-algebraic methods

Physical problems can often be formulated in terms of some algebraic framework, and this largely facilitates their discussion, because a number of results

can be directly interpreted in terms of the powerful machinery of group theory. In this case the ladder and weight operators typically appear as the elements of various algebras, while the Hamiltonian is constructed from the same operators: it can be related to the Casimir operator of the same algebra, or it can be an element of the algebra. The basis states (generally bound states) are assigned to group representations. In some cases the states assigned to the same irreducible representation are bound levels belonging to the same problem either with different energies or degenerate in energy. In these cases we talk about spectrum generating algebras and degeneracy algebras, respectively [1, 2]. In the latter case the Hamiltonian commutes with the elements of the algebra. When all the states of a problem are interconnected by the elements of some algebra, it takes the name of dynamical algebra. Examples for this are so(4,2) for the Coulomb potential [1] and mp(6) for the isotropic three-dimensional harmonic oscillator [52]: the latter is the algebra of the metaplectic group, Mp(6), which is the covering group of the symplectic group Sp(6,R), containing the states with even number of oscillator quanta in one irreducible representation and the states with odd number of quanta in another one.

In most applications to quantum mechanical potentials the algebras considered were the somewhat trivial su(1,1)$\sim$so(2,1) [1, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 12, 63, 64]

$$[J_z, J_\pm] = \pm J_\pm \tag{50}$$

$$[J_+, J_-] = -2\ J_z\ , \tag{51}$$

or its compact version, su(2)$\sim$so(3), which can be obtained by the $J_\pm \to \mathrm{i}J_\pm$ transformation. The reason certainly is that the one-dimensional quantum mechanical potential problems are usually too "simple" to accommodate large algebraic structures as degeneracy or dynamical algebras, because they typically support non-degenerate bound states, except for radial problems, like the harmonic oscillator mentioned before, where degenerate states exist with different values of the orbital angular momentum l. However, there are only two potentials (the harmonic oscillator and the Coulomb potentials) for which exact solutions are known for any value of l, so for the remaining potentials the degeneracy and dynamical algebras cannot be formulated in general.

Nevertheless, there are ways to implement less trivial algebras for potential problems too (as so(2,2) [65, 66] or so(4) [65, 67]). One possibility is considering *different* systems connected by the elements of some algebra. This is the case with the potential algebra [61, 68]. This is somewhat similar to the degeneracy algebra in the sense that the elements of the algebra connect degenerate levels

which, however, belong to different Hamiltonians. Not surprisingly, the problems discussed in terms of the potential algebra context are essentially the same ones that can be approached using the factorization method and SUSYQM. The number of exactly solvable problems admitting a potential algebra is limited to some well-known (shape-invariant [18]) problems like the Pöschl–Teller and Morse potentials, for example. The ladder operators of the su(1,1) potential algebras related to these first examples can be recognized as the shift operators of type A and B problems in the factorization method [32, 34] and also as the operators A and $A^\dagger$ related to these problems in SUSYQM. Type A and B problems are displayed in table 1 as PI and LIII class potentials [10] related to the Jacobi and generalized Laguerre polynomials, respectively. The practical equivalence of the su(1,1) potential algebra (and its compact version, su(2)) with the SUSYQM approach to shape-invariant potentials has been discussed in [69], and later on in [70, 71].

A further interesting aspect of potential algebras is that whenever they are non-compact, scattering states can be treated on an equal footing with bound states: the former belong to the continuous unitary irreducible representations and the latter to the discrete unitary irreducible representations of the relevant algebras. Non-compact potential algebras are so(2,1)$\sim$su(1,1) assigned to the Morse and the Pöschl–Teller potentials [61], but later on the larger so(2,2) potential algebra was also introduced [72, 73]. In fact, so(2,2) has been identified as the algebra of the rather general Natanzon family [8] of solvable potentials [72, 73]. In relation with Natanzon potentials, the concept of the satellite algebra has also been proposed [74]. This is an so(2,1) algebra which connects the states of *different* Natanzon potentials, but the energies of these states are not degenerate, in contrast with potential algebras.

To summarize the results and to establish the notation, here we discuss a general differential realization of the su(1,1) algebra (including also su(2)), in which the Hamiltonian is related to the Casimir invariant

$$C_2 = -J_+J_- + J_z^2 - J_z \tag{52}$$

$$= -J_-J_+ + J_z^2 + J_z \ , \tag{53}$$

which in this realization takes the form of a second-order differential operator. The eigenstates of C_2 and J_z (with eigenvalues $j(j+1)$ and m, respectively) serve as a basis for the irreducible representations of the SU(1,1) group, and can be labelled by $|jm\rangle$. In contrast with the unitary irreducible representations of the compact SU(2) group, which are finite-dimensional, those of SU(1,1) are infinite-dimensional and come in several types. First, there are discrete unitary ireducible representation called the discrete principal series D_j^+, for

which

$$j = -\frac{1}{2} - \frac{p}{2}, \qquad (p = \ 0,\ 1,\ \ldots) \tag{54}$$

$$m = -j,\ -j+1,\ -j+2,\ \ldots \tag{55}$$

hold. Another discrete series is D_j^-, with opposite signs for j and m. Then there are the continuous principal series with

$$j = -\frac{1}{2} + \mathrm{i}k\ , \qquad (0 < k) \tag{56}$$

$$m = 0 \pm 1,\ \pm 2,\ \ldots \quad \text{or} \quad m = \pm\frac{1}{2},\ \pm\frac{3}{2},\ \ldots \tag{57}$$

In the potential group approach these discrete and continuous unitary irreducible representations were associated with the bound- and scattering-state solutions of the Pöschl–Teller and Morse potentials, for example [61]. Finally, there are the so-called supplementary series, for which

$$-\frac{1}{2} < j < 0, \qquad m = 0 \pm 1,\ \pm 2,\ \ldots \tag{58}$$

holds, but which did not play any role in the formulation of the potential group approach [61]. The representation labels discussed here ultimately appear in the coupling coefficients and the energy formula of the physical problem. In this respect it has to be noted that they are not necessarily restricted to integer or half-integer values (as in some of the formulae above): using projective representations [75] arbitrary real values are allowed for them.

Following Sukumar [60], we consider the realization of su(1,1) in terms of the first-order differential operators

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm h(x)\frac{\partial}{\partial x} \pm g(x) + f(x)J_z + c(x)\right) \tag{59}$$

and

$$J_z = -\mathrm{i}\frac{\partial}{\partial\phi}\ , \tag{60}$$

together with the following form of the basis states

$$|jm\rangle = \Psi_{jm}(x) = \mathrm{e}^{\mathrm{i}m\phi}\psi_{jm}(x)\ . \tag{61}$$

Here x is a spatial coordinate variable while ϕ is an auxiliary phase factor. It was shown that (51) is fulfilled, provided that the relations

$$f^2(x) - h(x)\frac{\mathrm{d}f}{\mathrm{d}x} = 1 \tag{62}$$

and

$$h(x)\frac{\mathrm{d}c}{\mathrm{d}x} - c(x)f(x) = 0 \tag{63}$$

hold. (Equation (50) is automatically satisfied by this construction.) In terms of this realization the Casimir operator has the form

$$\begin{aligned} C_2 &= h^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x} + h(x)\left(\frac{\mathrm{d}h}{\mathrm{d}x} + 2g(x) - f(x)\right)\frac{\mathrm{d}}{\mathrm{d}x} \\ &- \left(f(x)g(x) - g^2(x) - h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + c^2(x)\right) - 2c(x)f(x)J_z + (1 - f^2(x))J_z^2. \end{aligned} \tag{64}$$

An important observation is that the su(1,1) algebra remains intact under variable and similarity transformations defined by $x \to z(x)$ and $\Psi_{jm}(x) \to \frac{1}{v(x)}\Psi_{jm}(x)$, respectively. The only essential changes occur for $h(x)$ and $g(x)$ in the two cases:

$$h(x) \to h(x(z))\frac{\mathrm{d}z}{\mathrm{d}x} \tag{65}$$

$$g(x) \to g(x) + h(x)\frac{\mathrm{d}}{\mathrm{d}x}\ln v(x)\ . \tag{66}$$

It has to be noted that su(1,1) remains unchanged provided the the functions governing transformations (i.e. $z(x)$ and $v(x)$) do not depend explicitly on m, the eigenvalue of generator J_z. Obviously, these transformations are nothing but the algebraic version of the method discussed previously in subsection 2.1, and (64) can be chosen to be the Schrödinger equation or the second-order differential equation of some special function. To this end only the four functions appearing in (59) have to be chosen in an appropriate way. In fact, in order to obtain the Schrödinger equation, one further condition has to be imposed on these functions in addition to (59) and (60):

$$\frac{\mathrm{d}h}{\mathrm{d}x} + 2g(x) - f(x) = 0\ . \tag{67}$$

It can also be noted that the mathematical construction presented here can be used to recover potentials associated with the compact su(2) algebra as well. This requires only the redefinition of the functions $h(x)$, $f(x)$, $g(x)$ and $c(x)$ by multiplying them with the constant imaginary factor i (or $-$i). This operation changes the sign of the right-hand side of (62), but does not affect (63). The expression for the Casimir operator in (65) also remains valid in its present

form: the $h(x) \to \mathrm{i}h(x)$, ..., etc. transformation simply changes the sign of its terms except for the one J_z^2.

In [69] this algebraic transformation method was applied systematically to the differential equations of orthogonal polynomials to recover su(1,1) and su(2) algebras associated with the shape-invariant potentials. The algebras turned out to be potential or spectrum generating algebras. Furthermore, it was found that in some cases the generators form a compact su(2) algebra rather than a non-compact su(1,1) algebra considered originally. This is related to the nature of the individual potentials (i.e. whether the number of their bound states is infinite or not, or whether they have scattering states, etc.).

Potential algebras have been recovered for two classes of shape–invariant potentials: the LIII class (i.e. the Morse potential) and the PI class which contains five individual potentials (see table 1). A characteristic feature of these potentials is that the $h(x)$ function in (59) is a constant, and the differential form of Casimir operator (64) is proportional to the Schrödinger equation up to a constant. For the same reason the J_+ and J_- generators of these potential algebras are practically identical with the $A^\dagger$ and A ladder operators of SUSYQM in the sense that they have the same effect on them [69]. This was confirmed later also in [70, 71]. These potentials belonging to the PI and LIII shape-invariant class correspond to type A and B potentials in the factorization method [32] and a study based on the Lie theory of special functions [34]. See also [10] for the details.

Spectrum generating algebras have been recovered for the LI, HI classes (i.e. the harmonic oscillators in three and one dimensions) and for special (symmetric) cases of PI and PII potentials. A common feature of these algebras is that in contrast with the case of potential algebras, now the variable transformation results an $h(x)$ function which is different from a constant, and, consequently the Hamiltonian has more complicated structure and does not necessarily commute with the generators. Due to the same circumstances (i.e. $h(x) \neq const$) the SUSYQM ladder operators differ from J_+ and J_- in this case. The Coulomb problem in three dimensions, i.e. the LII (or type F [32, 34]) class turned out to be inaccessible with this method. This is because the variable transformation $z(x)$ depends explicitly on the n and l quantum numbers (or the m eigenvalue of the J_z generator), and this prevents the use of the present differential realization of the su(1,1) algebra in this case.

Some of the algebras recovered in [69] can be embedded into some larger algebras. This is the case with the Scarf II, the generalized Pöschl–Teller, the Morse potential [63] and the three–dimensional harmonic oscillator [1], for example. It was also shown [72, 73] that the Natanzon class potentials can be

associated with an so(2,2) algebra. It also has to be noted that the differential realization (59), (60) of the su(1,1) algebra is only one possibility, and there are others in which the parametrization is done without auxiliary phase variables [53, 1, 56, 62, 64].

2.4 $\mathcal{PT}$ symmetry of potentials

The most recent symmetry concept discussed here requires the invariance of the Hamiltonian under the $\mathcal{PT}$ operation, i.e. the simultaneous action of the $\mathcal{P}$ spatial and the $\mathcal{T}$ time reflection operations (the latter essentially being complex conjugation). For one-dimensional potentials of nonrelativistic quantum mechanics this invariance requires $(V(-x))^* = V(x)$. Therefore $\mathcal{PT}$ invariant potentials are typically complex, and their real component is even, while their imaginary component is odd function of x. Although this symmetry concept requires only the commutation of a single operator with the Hamiltonian, i.e.

$$\mathcal{PT}H(\mathcal{PT})^{-1} = \mathcal{PT}H\mathcal{PT} = H \ , \tag{68}$$

so mathematically it seems less sophisticated than supersymmetric and Lie-algebraic symmetry concepts, it has surprisingly far reaching consequances regarding the energy spectrum of the potential. In particular, it was found that despite being complex (i.e. non-Hermitian), these potentials often have real energy eigenvalues, and this unusual feature was associated with the invariance of the Hamiltonian under the $\mathcal{PT}$ operation.

In the majority of quantum mechanical problems the Hamiltonian of the system is Hermitian, and this requirement guarantees that the bound-state energy eigenvalues are real. In some cases, however, the physical situation is such that the application of non-Hermitian Hamiltonians is justified. This happens, for example, for complex potentials used mainly in nuclear physics and accounting for the absorption of incident particles. In these models the discrete energy eigenvalues become complex in general. This is clearly different from the properties of $\mathcal{PT}$ invariant potentials. One important difference with respect to the complex optical potentials applied in nuclear physics, for example, is that these potentials are radial ones (see [76]), while the $\mathcal{PT}$ symmetric potentials are defined on the full x axis or on a finite domain of it, but they are also often defined on various contours of the complex x plane.

Strangely enough, the first examples for complex potentials with real spectra were found using numerical techniques [5]. The potentials considered in the first studies were typically polynomial type potentials with imaginary coupling coefficients [5]. These problems were defined on the complex x plane, and it

was found that normalizable solutions can be found along trajectories falling in certain wedges of the plane. Similar problems have been identified (sometimes in retrospect) from methods based on Fourier transformation analyses [77], semiclassical estimates, [78], numerical calculations [79], Sturm-Liouville-like theory [80], variational techniques [81] or perturbation methods [82].

Obviously, exactly solvable examples can be rather useful in the more thorough understanding of $\mathcal{PT}$ symmetric quantum mechanics, so exact analytical solutions to such problems have soon been derived [83, 84, 85, 86, 87, 88, 89, 90, 91, 92]. Most of the exactly solvable $\mathcal{PT}$ symmetric potentials have analogues in usual quantum mechanics. In some cases $\mathcal{PT}$ invariance is reached by simply setting the coupling constants of the odd potential terms to imaginary values. This was easy with potentials defined originally as one-dimensional problems on the full x axis [83, 84, 85]. In some other cases the coordinate x is shifted with an imaginary constant to $x - \mathrm{i}\epsilon$. One important aspect of this imaginary coordinate shift was that it cancelled the singularities typically appearing in some potentials at $x = 0$ (like the centrifugal barrier), and then these originally radial problems could be naturally extended to the full x axis [77, 86, 87]. For another class of potentials asymptotically deformed integration paths are defined to secure normalizability of the solutions [88, 89, 90, 91, 92]. It is not surprising that these exotic complex potentials had some unusual features. The cancellation of singularities encountered in the Hermitian versions of these potentials, for example, led to less strict boundary conditions, and thus to a richer energy spectrum.

It was also noticed that $\mathcal{PT}$ symmetry is neither a necessary, nor a sufficient condition for having real energy spectrum in a complex potential. It is not a necessary condition, because there are complex non-$\mathcal{PT}$ symmetric potentials with these properties [83, 90] some of these are complex supersymmetric partners of real potentials. Neither is $\mathcal{PT}$ symmetry a sufficient condition, because complex-energy solutions of such potentials are also known, and since in this case the energy eigenfunctions cease to be eigenfunctions of the PT operator, this scenario has been interpreted as the spontaneous breakdown of $\mathcal{PT}$ symmetry [5]. No general condition has been found for the breakdown of $\mathcal{PT}$ symmetry, but it has been observed that it usually characterizes strongly non-Hermitian problems [5, 93, 94].

The lack of Hermiticity raised questions about the probabilistic interpretation of the wavefunctions (probability density, continuity equation), and in general, about the definition of the norm and the inner product of the eigenvectors of the non-Hermitian Hamiltonian. It has been suggested, for example, that the $\psi^2(x)$ quantity should replace $|\psi(x)|^2$ in the definition of the

norm [81]. For unbroken $\mathcal{PT}$ symmetry this expression coincides with the $\psi(x)\psi^*(-x)$ quantity used in the definition of the pseudo-norm [95], which is obtained from the modified inner product $\langle\psi_i|\mathcal{P}|\psi_j\rangle$. This redefinition of the inner product was found to lead to the orthogonality of the energy eigenstates, but it also resulted in an indefinite metric, replacing the usual Hilbert space with the Krein space [96]. Efforts have been made to restore the Hermitian formalism using projection techniques [95, 96].

More recently $\mathcal{PT}$ symmetric quantum mechanics was put into a more general context, as the special case of pseudo-Hermiticity [97]. A Hamiltonian is said to be η-pseudo-Hermitian if

$$H^\dagger = \eta H \eta^{-1} \tag{69}$$

holds, where $\dagger$ denotes the adjoint operation. (Sometimes pseudo-Hermiticity is defined by the intertwining relation $H^\dagger\eta = \eta H$, because this doesn't require the invertibility of the η operator.) Obviously, $\eta = 1$ recovers conventional Hermiticity, while $\mathcal{PT}$ symmetric Hamiltonians are $\mathcal{P}$-psudo-Hermitian. It was also shown, that for η-pseudo-Hermitian systems the inner product has to be redefined as

$$\langle\psi_1|\psi_2\rangle_\eta \equiv \langle\psi_1|\eta|\psi_2\rangle \ , \tag{70}$$

which recovers the conventional inner product for $\eta = 1$ and the one discussed previously for $\eta = \mathcal{P}$. Based on these general arguments it was demonstrated [97] that a Hamiltonian is pseudo-Hermitian if and only if its eigenvalues are real or come in complex conjugate pairs, as was the observation for $\mathcal{PT}$ symmetric potentials. With these general considerations an explanation was given for the existence of those non-Hermitian problems which have real spectra, but do not possess $\mathcal{PT}$ invariance. In fact, the term psudo-Hermiticity has already been introduced long before the formulation of $\mathcal{PT}$ symmetric quantum mechanics [98, 99].

Since the evolution of $\mathcal{PT}$ symmetric quantum mechanics is not yet finished, some recent results will be presented in subsection 3.4, in relation with my activity in the field.

3 Results

The relatively simple shape-invariant potentials have already been analyzed thoroughly in terms of supersymmetric quantum mechanics and algebraic approaches, so my results in subsections 3.1, 3.2 and 3.3 concern mainly more general potentials, or sometimes the non-standard treatment of shape-invariant potentials. $\mathcal{PT}$ symmetry is a more recent subject, so there was more room for illustrating the methods with the simple shape-invariant potentials. For this reason most of the examples I present in subsection 3.4 are related to shape-invariant potentials. This also holds for the examples presented in subsection 3.5, where I analyze the relation of the three symmetry concepts.

3.1 Some non-trivial solvable potentials

In this subsection I present illustrative examples for Natanzon and Natanzon confluent potentials introduced by various methods. Some of these are "implicit" potentials, i.e. the $z(x)$ function defining the variable transformation in terms of the method discussed in subsection 2.1 is given in the implicit $x(z)$ form only, while in some other cases the $z(x)$ function is known explicitly, but the energy eigenvalues have to be calculated from the roots of a cubic equation. Besides potentials introduced by me, I also discuss some others which I analyze in the forthcoming subsections.

3.1.1 An "implicit" type Natanzon confluent potential: the generalized Coulomb problem

The generalized Coulomb potential was introduced previously [16] as a radial potential in three spatial dimensions, while here I discuss it in arbitrary dimensions and analyze some of its specific properties. This potential contains both the Coulomb and the harmonic oscillator potentials (of various dimensions) as special (shape-invariant) limits, and establishes a novel type of Coulomb-oscillator connection. This follows from the substitution of the generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ into (6) as the $F(z)$ special function, i.e. writing $Q(z) = -1 + (\alpha+1)/z$ and $R(z) = n/z$ [19]:

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{z(x)}\left(n + \frac{\alpha+1}{2}\right) - \frac{(z'(x))^2}{4} - \frac{(z'(x))^2}{z^2(x)}\left(\frac{\alpha^2-1}{4}\right) . \tag{71}$$

If we identify the third, fourth and fifth term on the right-hand side with the (energy) constant, then we arrive at differential equations defining the harmonic oscillator (LI), Coulomb (LII) and Morse (LIII) potentials [10, 16]. (See table 1.) The generalized Coulomb potential is obtained if the linear combination of the third and the fourth term is identified with a constant. Introducing for convenience the $z(r) = \rho h(r)$ notation, the differential equation defining this potential is $(h')^2 = Ch(h+\theta)^{-1}$, which is solved implicitly by [16]

$$r = r(h) = C^{-\frac{1}{2}} \left[\theta \tanh^{-1} \left(\left(\frac{h}{h+\theta} \right)^{\frac{1}{2}} \right) + (h(h+\theta))^{\frac{1}{2}} \right] \ . \tag{72}$$

The $h(r)$ function maps the $[0, \infty)$ half axis onto itself and can be approximated with $h(r) \simeq C^{\frac{1}{2}} r$ and $h(r) \simeq Cr^2/(4\theta)$ in the $r \to \infty$ and $r \to 0$ limits, respectively.

Adapting the notation to D spatial dimensions, the generalized Coulomb potential is [P5]

$$\begin{aligned} V(r) \ &= \ -\frac{1}{r^2} \left(l + \frac{D-3}{2} \right) \left(l + \frac{D-1}{2} \right) + \left(\beta - \frac{1}{2} \right) \left(\beta - \frac{3}{2} \right) \frac{C}{4h(r)(h(r)+\theta)} \\ & -\frac{q}{h(r)+\theta} - \frac{3C}{16(h(r)+\theta)^2} + \frac{5C\theta}{16(h(r)+\theta)^3} \ , \end{aligned} \tag{73}$$

where the first term compensates the centrifugal term in the Schrödinger equation (with units of $\hbar = 2m = 1$)

$$H_0 \psi(r) \equiv \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r^2} (l + \frac{D-3}{2})(l + \frac{D-1}{2}) + V(r) \right) \psi(r) = E\psi(r) \ . \tag{74}$$

The centrifugal term originates from the kinetic term, i.e. from the D-dimensional Laplace operator after separating the angular variables. The bound-state wavefunctions solving (74) are normalized as

$$\int_0^\infty |\psi(r)|^2 \mathrm{d}r = 1 \ . \tag{75}$$

Bound states are located [P5] at

$$E_n = -\frac{C}{4} \rho_n^2 \ , \tag{76}$$

where

$$\rho_n = \frac{2}{\theta} \left(\left((n + \beta/2)^2 + \frac{q\theta}{C} \right)^{\frac{1}{2}} - (n + \beta/2) \right) \ . \tag{77}$$

The bound-state wavefunctions can be written in terms of generalized Laguerre polynomials as

$$\begin{aligned}\psi_n(r) &= C^{\frac{1}{4}}\rho_n^{\frac{\beta+1}{2}}\left(\frac{\Gamma(n+1)}{\Gamma(n+\beta)(2n+\beta+\rho_n\theta)}\right)^{1/2} \\ &\quad\times(h(r)+\theta)^{\frac{1}{4}}(h(r))^{\frac{2\beta-1}{4}}\exp(-\frac{\rho_n}{2}h(r))L_n^{(\beta-1)}(\rho_n h(r))\ . \qquad (78)\end{aligned}$$

Potential (73) clearly carries angular momentum dependence: its first term merely compensates the centrifugal term arising from the kinetic term of the Hamiltonian. Its second term also has r^{-2}-like singularity (due to $h^{-1}(r)$), and as it will be demonstrated later, it cancels the angular momentum dependent term in the two important limiting cases that recover the D-dimensional Coulomb and the harmonic oscillator potentials. The third term of (73) represents an asymptotically Coulomb-like interaction, while the remaining two terms behave like r^{-2} and r^{-3} for large values of r.

These general results can readily be specialized to $D = 3$ and $l = 0$, i.e. for the usual Coulomb potential. The S matrix of the generalized Coulomb potential can be derived in complete analogy with that of the Coulomb problem for $D = 3$. Although this can only be done exactly for $l = 0$, the singular term imitating the centrifugal term in (73) can be defined to be part of the potential. Following the method of [101], the S matrix for $l = 0$ is expressed as [P5]

$$S_0(k) = (-1)^{\frac{\beta}{2}+1}\frac{\Gamma(\frac{\beta}{2}+\mathrm{i}\nu)}{\Gamma(\frac{\beta}{2}-\mathrm{i}\nu)}\ , \qquad (79)$$

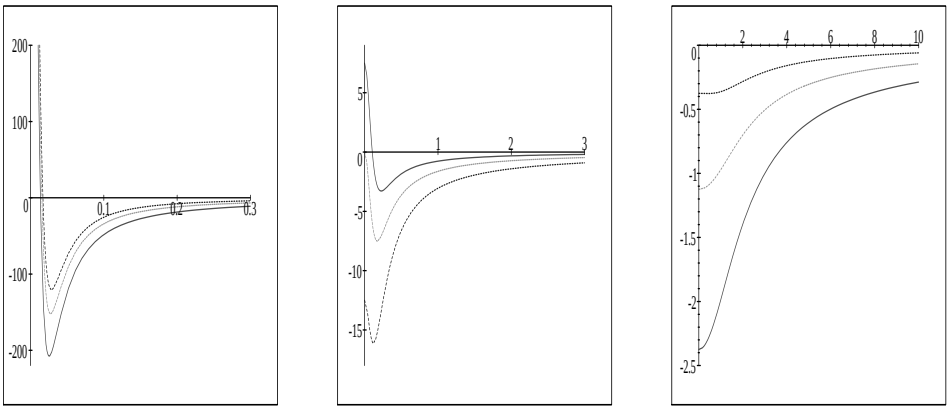
where

$$\mathrm{i}\nu \equiv \mathrm{i}\nu(k) = \frac{\rho\theta}{4} - \frac{q}{C\rho}\ , \qquad \rho \equiv \rho(k) = -\frac{2\mathrm{i}k}{C^{1/2}}\ . \qquad (80)$$

The extra phase factor in $S_0(k)$ appears because of the r^{-2}-type singular term, which is now defined to be part of the potential. In the Coulomb limit this expression becomes part of the centrifugal term, which is dealt with separately.

The long-range behavior of potential (73) suggests its use in problems associated with the electrostatic field of some charge distribution. The deviation from the Coulomb potential close to the origin can be viewed as replacing the point-like charge with an extended charged object. The relevant charge density is readily obtained [P5] from the potential using

$$\rho(r) = -\frac{1}{4\pi\mathrm{e}}\Delta v(r)\ . \qquad (81)$$



$\theta = 0.01$ $\theta = 0.1$ $\theta = 1$

Figure 2: The generalized Coulomb potential for q=0.5, 1.25, 2.5; θ=0.01, 0.1, 1; C=1 and $\beta = 3/2$. $l = 0$ and $D = 3$ is also implied. In each panel the largest q corresponds to the lowest curve. Note the different scales of the horizontal (r) and the vertical ($V(r)$) axes.

In figure 2 the actual shape of potential (73) is plotted for various parameter sets. This potential is suitable for describing the Coulomb field of extended objects. It is a general feature of potential (73) that for small values of θ a (finite) positive peak appears near the origin, which also manifests itself in a repulsive "hard core", corresponding to a region with positive charge density [P5].

The generalized Coulomb potential is a Natanzon confluent potential by construction, and it contains the Coulomb and the harmonic oscillator potentials as special cases. This is also reflected by the structure of the function $h(r)$ in (72): when $h(r)$ is proportional to r and r^2, one obtains the Coulomb problem and the harmonic oscillator potential, respectively. These limits can readily be realized by specific choices of the parameters in (72): the first one follows from the $\theta \to 0$ limit [16], while the second one is reached by taking $\theta \to \infty$, while keeping $C/\theta \equiv \tilde{C}$ constant [100, P5].

Besides taking the $\theta \to 0$ limit, the Coulomb problem in D-dimensions is recovered from (73) by the $\beta = 2l + D - 1$ and $C^{-\frac{1}{2}}q = Ze^2$, choices: the third term of (73) becomes the Coulomb term, the fifth one vanishes, while the other three all become proportional with r^{-2} and cancel out completely.

In order to reach the oscillator [P5] limit one also has to redefine the potential (73) and the energy eigenvalues by adding q/θ to both. This choice

simply represents resetting the energy scale: $E = 0$ corresponds to $V(r \to \infty)$ for the Coulomb problem, and to $V(r = 0)$ for the harmonic oscillator. (Note that the energy eigenvalues also have different signs in the two cases.) Besides $C/\theta = \tilde{C}$, the $\tilde{q} \equiv q/\theta^2$ parameter also has to remain constant in the $\theta \to \infty$ transition here. The potential thus adapted to the harmonic oscillator limit reads

$$\begin{aligned}\tilde{V}(r) \;\equiv\; & V(r) + q\theta = \\ & -\frac{1}{r^2}\left(l + \frac{D-3}{2}\right)\left(l + \frac{D-1}{2}\right) + \left(\beta - \frac{1}{2}\right)\left(\beta - \frac{3}{2}\right)\frac{\tilde{C}}{4h(r)(1+\frac{h(r)}{\theta})} \\ & -\frac{\tilde{q}h(r)}{1+\frac{h(r)}{\theta}} - \frac{3\tilde{C}}{16\theta}\frac{1}{\left(1+\frac{h(r)}{\theta}\right)^2} + \frac{5\tilde{C}}{16\theta}\frac{1}{\left(1+\frac{h(r)}{\theta}\right)^3} \;. \end{aligned} \tag{82}$$

The harmonic oscillator potential is recovered from (82) by the $\beta = l + D/2$ and $\tilde{C}\tilde{q} = \omega^2$ choice. The two last terms in (82) vanish, the first and the second cancel out, while the third one reproduces the harmonic oscillator potential. The new form of the energy eigenvalues is

$$\tilde{E}_n \equiv E_n + q/\theta = \tilde{C}(2n+\beta)\left[\left(\frac{1}{\theta^2}(n+\frac{\beta}{2})^2 + \frac{\tilde{q}}{\tilde{C}}\right)^{\frac{1}{2}} - \frac{1}{\theta}(n+\frac{\beta}{2})\right] \;, \tag{83}$$

which indeed, reduces to the $\tilde{E}_n = (2n + l + D/2)\omega$ oscillator spectrum in the $\theta \to \infty$ limit. The wavefunctions (78) are unchanged, except for the redefinition of the parameters.

Note that the generalized Coulomb problem establishes a link between the Coulomb problem and the harmonic oscillator potential in different spatial dimensions. This is best seen by inspecting the wavefunctions (78). If the $\beta - 1$ parameter of the generalized Laguerre polynomial is required to be the same in the two limits, we get an interrelation between the value of the angular momentum and the spatial dimension to be used for the Coulomb and the harmonic oscillator case:

$$l^O + \frac{D^O}{2} = 2l^C + D^C - 1 \;. \tag{84}$$

Considering $l^O = 2l^C$, (84) implies $D^O = 2D^C - 2$ which establishes link between the (D^C, D^O) =(2,2), (3,4), (4,6), (5,8), ... etc. pairs. (Of these the (3,4) pair corresponds to the Kustaanheimo-Steifel transformation [102].) This case is called the "direct map" between the Coulombic and oscillator

solutions in [103], while with the $l^O = 2l^C + \lambda$ choice the "general map" can be recovered. These results suggest that the generalized Coulomb potential can be used to formulate a continuous transition between the Coulomb and harmonic oscillator potentials, as opposed to the usual procedure that employs a unique variable and parameter transformation to reach this goal.

The formalism developed previously is valid for $D = 1$ too, nevertheless, some particular properties of one-dimensional problems justify a separate treatment of this case [P5]. First, the implicit definition of the $h(r)$ function in (72) has to be extended to negative values of r, which we now denote with x. Using the notation of (72), we can write that $x = r(h)$ for $x \geq 0$ and $x = -r(h)$ for $x < 0$. The normalization of the wavefunctions in (78) also has to be modified with a factor of $2^{-1/2}$, accounting for the fact that the integration now runs from $-\infty$ to ∞.

For one-dimensional problems it is natural to set l to 0 besides $D = 1$, which eliminates the centrifugal term in (73). Furthermore, in order to avoid r^{-2}-like singularities at $x = 0$ the second term in (73) also has to be canceled by setting β either to $1/2$ or to $3/2$. Elementary calculations show that the latter choice corresponds to bound-state wavefunctions that vanish at $x = 0$, and essentially represent physical solutions of the problem in higher dimensions as well, while the former choice recovers solutions that do not vanish in general at $x = 0$. These two possibilities can naturally be interpreted as odd and even solutions of the one-dimensional potential problem. Furthermore, for $x \geq 0$ the two types of wavefunctions can be rewritten into a common notation (up to a sign) by making use of the relation of generalized Laguerre and Hermite polynomials, when the former ones have $\alpha = 1/2$ or $\alpha = -1/2$ as parameters [19]:

$$\begin{aligned} \psi_N^{(D=1)}(x) &= \frac{C^{\frac{1}{4}} \tilde{\rho}_N^{\frac{3}{4}}}{2^N \left(\Gamma(\frac{N+1}{2})\Gamma(\frac{N}{2}+1)(N+\frac{1}{2}+\theta\tilde{\rho}_N\right)^{\frac{1}{2}}} \\ &\quad \times (h(x)+\theta)^{\frac{1}{4}} \exp(-\frac{\tilde{\rho}_N}{2} h(x)) H_N((\tilde{\rho}_N h(x))^{\frac{1}{2}}) \ . \end{aligned} \tag{85}$$

For $x \leq 0$ the bound-state wavefunctions satisfy

$$\psi_N^{(D=1)}(-x) = \begin{cases} \psi_N^{(D=1)}(x) & \text{for } N = 2n \ (\beta = \frac{1}{2}) \\ -\psi_N^{(D=1)}(x) & \text{for } N = 2n+1 \ (\beta = \frac{3}{2}) \ . \end{cases} \tag{86}$$

In (85) we defined $\tilde{\rho}_N$ as

$$\tilde{\rho}_N \equiv \frac{1}{\theta}\left[\left((N+\frac{1}{2})^2+4\frac{q\theta}{C}\right)^{\frac{1}{2}}-(N+\frac{1}{2})\right] \ , \tag{87}$$

which reduces to $\rho_{[\frac{N}{2}]}$, where even and odd values of N have to be chosen with $\beta = 1/2$ and $\beta = 3/2$, respectively, and the integer part of $N/2$ corresponds to n used in ρ_n in (77).

An interesting aspect of this potential is that it remains finite at $x = 0$ ($V(0) = -q/\theta + C/(8\theta^2)$) for any finite value of θ, however, a narrow, finite peak appears in the $\theta \to 0$ limit, which then becomes an attractive $-3/(16r^{-2})$-like singularity in the Coulomb limit. This is due to the fourth term in (73) and it corresponds to a "weak" singularity in the sense that the center of attraction is not strong enough for the particle to become infinitely bound [104]. This finite barrier arising for small, but finite θ values also introduces the possibility of studying tunneling effects in symmetric potential wells. We also note that besides being finite at $x = 0$, potential (73) has continuous derivative there, as can directly be verified.

Based on these features the $D = 1$ version of potential (73) can be used to analyze the peculiarities of the one-dimensional Coulomb potential defined as $V^c = -\mathrm{e}^2/|x|$. This problem has been the subject of intensive studies in the past couple of decades, but there is still some controversy in the interpretation of the results (see e.g. [105] for a recent review). The unusual features attributed to this singular problem include degenerate eigenvalues [106] interpreted in terms of a hidden O(2) symmetry [107], an infinitely bound ground state [106] and continuous bound-state spectrum [108]. The last two of these were later found to be based on unacceptable solutions of the Schrödinger equation [109, 110], while the unexpected degeneracy was explained by an impenetrable barrier at $x = 0$, which separates the problem into two disjoint, non-communicating systems with identical energy spectra [110] and makes even the concept of parity obsolete here [111]. Most authors discussing the one-dimensional Coulomb problem agree that the usual techniques of quantum mechanics alone in dealing with potentials are not sufficient in this case. In [112] for example self-adjoint extension of the relevant differential operator has been discussed.

The one-dimensional version of potential (73) can be chosen in such a way that it becomes close to non-singular potentials used in the approximation of the true one-dimensional Coulomb potential. In fact, with appropriate choice of q and θ any desired Coulomb asymptotics and $V(x = 0)$ value can be generated. Figure 2 shows potentials with rounded-off shape near $x = 0$ ($\theta = 1$,

$q = 0.5$) and also ones close to the Coulomb potential with a cutoff $-\mathrm{e}^2/(|x|+a)$ ($\theta = 1$, $q = 2.5$). In contrast with these modified Coulomb potentials, all calculations can be performed exactly with (73). In the $\theta \to 0$ limit the generalized Coulomb potential recovers the one-dimensional Coulomb potential supplemented with the $-\frac{3}{16}x^{-2}$ term. This means that the one-dimensional Coulomb potential cannot be reached exactly, nevertheless, reasonable approximations of it can be given.

The odd solutions, of course, vanish at $x = 0$, while the even solutions have non-zero value there as long as $\theta > 0$ holds. In the $\theta \to 0$ limit $\psi^{(D=1)}_{N=2n}(0)$ varies with $\theta^{1/4}$, so the even solutions also tend to zero at $x = 0$. This is in accordance with the behaviour of "weakly attractive" γr^{-2} type singular potentials on the half line: for $-\frac{1}{4} < \gamma < 0$ both independent solutions vanish at the origin, so the wavefunctions are necessarily zero at $r = 0$. If we try to extend the $N = 2n$ solutions (85) in the $\theta = 0$ Coulomb limit to the $(-\infty, 0)$ domain we find that due to their $x^{1/4}$ type behavior at the origin the derivative of an even wavefunction would not be continuous anymore.

The reflexion and transmission coefficients can be analyzed using the asymptotic behavior of the general solutions of the one-dimensional problem. These can be chosen to be even and odd functions of x. The even and the odd solutions can be defined for $x \geq 0$, setting $\beta = \frac{1}{2}$ and $\frac{3}{2}$, respectively; and their extension to $x \leq 0$ can be given using a formula similar to (86). The two solutions are interrelated by equation 6.3(3) of [114]. Due to the symmetric nature of the one-dimensional potential ($V(x) = V(-x)$) it is enough to analyze the asymptotic behavior of the solutions for $x \to \infty$: the $x \to -\infty$ case follows naturally. Straightforward calculations show that the reflexion coefficient is

$$R(k) = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2}\left(\frac{\Gamma(\frac{1}{4}+\mathrm{i}\nu)}{\Gamma(\frac{1}{4}-\mathrm{i}\nu)} - \mathrm{i}\frac{\Gamma(\frac{3}{4}+\mathrm{i}\nu)}{\Gamma(\frac{3}{4}-\mathrm{i}\nu)}\right) . \tag{88}$$

Strong reflexion is found for potentials having a (finite) barrier in $x = 0$ (like those in figure 2 with $\theta = 0.01$), while more regular shapes (like that in figure 2 with $\theta = 1$ and $q = 2.5$, for example) give weak reflexion. Our findings seem to support the existence of the space splitting effect [111] valid for the Coulomb potential on one dimension [P5].

> Besides its mathematical aspects the one-dimensional Coulomb potential has physical relevance too, in the description of the hydrogen atom in strong magnetic field [115], for example. In such practical calculations it is reasonable to use a non-singular Coulomb-like potential instead of the true one-dimensional Coulomb potential: the finite size of the nucleus can be a justification for this.

This means that the one-dimensional Coulomb potential might not be sufficient in such calculations: the basis defined with it simply does not contain even-parity states. In practical calculations therefore the use of bases like that assigned to the generalized Coulomb potential is necessary.

3.1.2 Some "implicit" type Natanzon potentials

Here I mention some specific features of Natanzon potentials constructed by me, and also describe briefly some known ones which I am using in subsections 3.2 and 3.3. Natanzon potentials can conveniently be generated in terms of the transformation method discussed in subsection 2.1 by identifying $F(z)$ with a Jacobi polynomial: $F(z) = P_n^{(\alpha,\beta)}(z)$. Equation (7) is an explicit form for $E - V(x)$ in this case:

$$\begin{aligned} E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1-z^2(x)} n(n+\alpha+\beta+1) \\ &+ \frac{(z'(x))^2}{(1-z^2(x))^2}\left[\frac{1}{2}(\alpha+\beta+2) - \frac{1}{4}(\beta-\alpha)^2\right] \\ &+ \frac{(z'(x))^2 z(x)}{(1-z^2(x))^2}\frac{1}{2}(\beta-\alpha)(\beta+\alpha) \\ &+ \frac{(z'(x))^2 z^2(x)}{(1-z^2(x))^2}\left[\frac{1}{4} - \left(\frac{\alpha+\beta+1}{2}\right)^2\right] \quad . \end{aligned} \tag{89}$$

As discussed in [10], one selects differential equations of the type (9) for $z(x)$ to get constant terms on the right-hand side of (89). In [10] the first two nontrivial terms were picked, leading to the PI and PII potential classes. The defining differential equation of these is $(z')^2(1-z^2)^{-1} = C$ and $(z')^2(1-z^2)^{-2} = C$. Later in [15] the third "PIII" possibility, $z(z')^2(1-z^2)^{-2} = C$ was also discussed, resulting in an "implicit potential" with no shape-invariant limit. Further choices of the defining (9) differential equation yields further Natanzon potentials, and we are going to analyze some of them here. Obviously, there can be only three linearly independent terms of the type $(z')^2 z^\kappa (1-z^2)^2$ (89), with κ =0, 1 and 2 as the most convenient choice, so the last term in (89) is redundant. However, for convenience and later use we keep it.

Considering the differential equation [P1]

$$\frac{(z')^2 z}{(1-z^2)(1-z)} = C \tag{90}$$

obviously corresponds to taking a special linear combination of the third and fourth term in (89). For $C = -1$ (90) is solved by the implicit

$$x(z) = 2\tan^{-1}[(-z^{-1}-1)^{1/2} - 2^{1/2}\tan^{-1}\left[[(-z^{-1}-1)/2]^{1/2}\right] \qquad (91)$$

function, which maps $z \in [-1, 0]$ to the finite $x \in [0, \pi(1-2^{-1/2})]$ domain. The resulting potential

$$V(x(z)) = -\frac{B}{z} - \frac{4z^4+4z^3-z^2-2z-5}{16z^3(1+z)} \qquad (92)$$

is then also defined on a finite range, similarly to the trigonometric shape-invariant potentials. The energy eigenvalues are $E_n = \alpha_n^2/2 - B$, where B and α_n are defined by

$$B = \frac{\alpha_n^2}{2} - \left(n + \frac{\alpha_n}{2}\right)\left(n + \frac{\alpha_n}{2} + 1\right) \qquad (93)$$

and the bound-state wavefunctions are

$$\psi_n(x(z)) \sim z^{1/4}(1+z)^{1/4}(1-z)^{\alpha_n/2}P_n^{(\alpha_n,0)}(z)\ . \qquad (94)$$

Equation (93) requires $B \geq -1/4$.

A close inspection of the behaviour of this potential near the endpoints reveals that it is singular and attractive, but the $-\gamma/x^2$-type singularities do not belong to the prohibitive $\gamma > 1/4$ domain, where the particle falls into the center of attraction [104]. A particularly interesting case arises for $B = 1/4$, when the square root in (93) disappears and the energy eigenvalues become *identical* with those of the potential $V(x) = c^2 s(s-1)\mathrm{cosec}^{\,2}(cx) - 1/4$ with $s = 1/2$ and $C = 2 + 2^{1/2}$. This is a downward oriented symmetric Scarf I or Pöschl–Teller I potential (see table 1), which is attractive, but its singularity is in the "weakly attractive" domain ($\gamma = 5/36$), so its solutions are still physical [116]. Therefore it can either be considered an elementary cell of a periodic potential or a single finite-range potential, similarly to the trigonometric shape-invariant potentials.

Finally, we note that potential (92) can be obtained from the general Natanzon potentials (11) by taking $a_1 = -4$, $c_1 = 0$, $c_0 = -2$, $f = -1$, $h_0 = 2B - 3/2$ and $h_1 = 1$ [P1].

As another illutrative example we can choose the defining differential equation [P18]

$$\frac{(z')^2(z+\gamma)^2}{(1-z^2)^2} = C\ , \qquad (95)$$

which is solved by

$$x(z) = \ln\left[(1+z)^{\frac{1+\gamma}{2}}(1-z)^{\frac{1-\gamma}{2}}\right] \tag{96}$$

for $C = 1$, which merely corresponds to setting the length scale to a particular value. This is an implicit function mapping $z \in [-1, 1]$ to $x \in (-\infty, \infty)$, and it resembles the $z(x) = \tanh x$ function, except that its shape is set by γ.

Rearranging the terms in (89) so that n appears only in the constant term we find that the α and β parameters of the Jacobi polynomials have to be related by $\beta = \alpha(\gamma-1)/(\gamma+1)$ unless $\gamma = 1$ holds. Furthermore, α picks up n-dependence, since it has to fulfil the condition $\alpha^2 = [n + \gamma(\alpha+n)][(n+1+\gamma(\alpha+n+1)] - A(\gamma+1)^2$. Then the resulting potential takes the form

$$V(x(z)) = -\frac{A(1-z^2)}{(z+\gamma)^2} - \frac{z(1-z^2)}{(z+\gamma)^3} - \frac{3(1-z^2)^2}{4(z+\gamma)^4} \ . \tag{97}$$

The energy eigenvalues are $E_n = -\alpha_n^2/(\gamma+1)^2$ and the corresponding bound-state wavefunctions become

$$\psi_n(x(z)) \sim (z+\gamma)^{1/2}(1-z)^{\alpha_n/2}(1+z)^{\beta_n/2}P_n^{(\alpha_n,\beta_n)}(z) \ , \tag{98}$$

where

$$\alpha_n = \left[-\gamma(2n+1) + [4n(n+1) + 4A(\gamma^2-1) + \gamma^2]^{\frac{1}{2}}\right][2(\gamma-1)]^{-1} = \beta_n\frac{\gamma+1}{\gamma-1}. \tag{99}$$

It can also be established that the number of bound states is limited by $n < -1/2 + (A+1/4)^{1/2}$, so it is independent of γ.

The potential is essentially single hole shifted with respect to $x = 0$, its depth is set by A, while γ changes its shape in such a way that making it deeper goes with making it narrower and vice versa, in accordance with the observation the γ does not influence the number of states.

Finally, we note that potential (97) is a Natanzon potential (11) with $a_1 = 4$, $c_1 = \gamma^2 + 3$, $c_0 = (\gamma+1)^2$, $f = -4A$, $h_0 = h_1 = 1$ [P18]. It also has to be mentioned that radically different results are obtained for $\gamma = 1$, which corresponds to a potential discussed in subsection 3.1.3. This particular choice of γ even changes this "implicit" potential into an "explicit" one, because the (95) differential equation becomes explicitly solvable for z.

In order to recover the Ginocchio potential [13] one can consider (89) with $\alpha = \beta$, in which case the Jacobi polynomials reduce to the simpler Gegenbauer polynomials [19] in (89) only two essential terms remain with $(z')^2(1-z^2)^{-1}$ and $(z')^2(1-z^2)^{-2}$. These two terms lead to PI and PII type potentials in

table 1, but only with their restricted (symmetric) version due to the $\alpha = \beta$ choice. Actually, these PI and PII potentials coincide pairwise and result in the hyperbolic and trigonometric versions of the Pöschl–Teller potential hole.

The Ginocchio potential can be obtained by combining the surviving two terms by setting [C3]

$$(z')^2 = C(1-z^2)^2(\delta+1-z^2)^{-1} \ . \tag{100}$$

This contains the above two cases if $\delta = 0$ and $\delta \to \infty$ is chosen (here also prescribing $C\delta^{-1} \to \widetilde{C} = finite$). Equation (100) is solved, for example, by the implicit $x(z)$ function

$$C^{\frac{1}{2}}x = \tan^{-1}\left[z(\delta+1-z^2)^{-\frac{1}{2}}\right] + \delta^{\frac{1}{2}}\tanh^{-1}\left[\delta^{\frac{1}{2}}z(\delta+1-z^2)^{-\frac{1}{2}}\right], \tag{101}$$

which, up to some variable and parameter transformation, is the equation defining the Ginocchio potential [13]. $E - V(x)$ in (7) now takes the form

$$E - V(x) = C\left((n+\alpha)^2 - \frac{4\Delta+3-3\delta}{4M(z)} - \frac{3\delta(3\delta+2)}{4M^2(z)} + \frac{5\delta^2(\delta+1)}{4M^3(z)}\right), \tag{102}$$

$$\Delta \equiv \left(\alpha+\frac{1}{2}\right)\left(\alpha-\frac{3}{2}\right) + \delta\left(n+\alpha+\frac{1}{2}\right)\left(n+\alpha-\frac{1}{2}\right), \tag{103}$$

where we used the notation $M(z) \equiv \delta + 1 - z^2$. Equations (102) and (103) as well as the wavefunctions

$$\psi_n(x) = (\delta+1-z^2(x))^{\frac{1}{4}}(1-z^2(x))^{\frac{1}{2}(\alpha-\frac{1}{2})}C_n^{(\alpha)}(z(x)) \tag{104}$$

reduce to the corresponding ones in [13] for $\delta = (\lambda^2-1)^{-1}$, $C = \lambda^4(\lambda^2-1)^{-1}$ and $\alpha = \mu + \frac{1}{2}$. We are going to present an su(1,1) algebra related to the Ginocchio potential in subsection 3.3.1.

Just as the Ginocchio potential [13] contains the symmetric Pöschl–Teller potential in a special limit, the generalized Ginocchio potential [14] contains the generalized Pöschl–Teller potential as a special case. The main difference between the latter and former potentials is that the latter ones have an r^{-2} type singularity at $r = 0$ and are interpreted as radial potentials. Since we are going to analyze the generalized Ginocchio potential in subsection 3.2.2, we present its essential properties here and discuss it as a member of the Natanzon potential class.

In its original formulation of the generalized Ginocchio potential was defined to describe a particle with an effective mass [14]. This effective (i.e.

coordinate-dependent) mass is clearly incompatible with the standard Schrödinger equation we consider, therefore here we take a special case allowing constant mass. In what follows we parametrize the generalized Ginocchio potential as

$$\begin{aligned} V_0(r) &= -\frac{\gamma^4}{\gamma^2+\sinh^2 u}\Big[s(s+1)+1-\gamma^2-\frac{5\gamma^2(1-\gamma^2)^2}{4(\gamma^2+\sinh^2 u)^2} \\ &\quad -\frac{3(1-\gamma^2)(3\gamma^2-1)}{4(\gamma^2+\sinh^2 u)}-\lambda(\lambda-1)\coth^2 u\Big]\ , \end{aligned} \tag{105}$$

where we changed the notation of [14] to make it more suitable for our purposes. This form can be obtained from the original formulae by setting $a=0$ (which cancels the effective mass term), $\alpha_l=\lambda-\frac{1}{2}$, $\nu_l=s$, $\beta_{nl}=\mu$, $\lambda=\gamma$ and $y=\sinh u(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}$.

The (generalized) Ginocchio potential is an example for "implicit" potentials, because it is expressed in terms of a function $u(r)$ which is known only in the implicit $r(u)$ form:

$$\begin{aligned} r &= \frac{1}{\gamma^2}\Big[\tanh^{-1}\Big((\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\Big) \\ &\quad +(\gamma^2-1)^{\frac{1}{2}}\tan^{-1}\Big((\gamma^2-1)^{\frac{1}{2}}(\gamma^2+\sinh^2 u)^{-\frac{1}{2}}\sinh u\Big)\Big]\ , \end{aligned} \tag{106}$$

which is essentially the same as (101) using the new parametrization. Now r can take values from the positive half axis, which is mapped by the monotonously increasing implicit $u(r)$ function onto itself. This function is, actually, the solution of an ordinary first-order differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}r}=\frac{\gamma^2\cosh u}{(\gamma^2+\sinh^2 u)^{\frac{1}{2}}} \tag{107}$$

defining a variable transformation connecting the Schrödinger equation with the differential equation of the Jacobi (and Gegenbauer) polynomials. Considering the problem as a Natanzon potential, the $z=\cosh^{-2}u$ substitution has to be made in (11) to (14), and this sets the a_1, c_1 and c_0 parameters in (12) to $a_1=(1-\gamma^2)\gamma^{-4}$, $c_1=0$ and $c_0=\gamma^{-4}$. It can be seen from (106) and (107) that $u(r)$ behaves approximately as γr near the origin, and as $\gamma^2 r$ for large values of r. In the $\gamma\to 1$ limit u becomes identical with r, and (105) reduces to the generalized Pöschl–Teller potential, which is in line with the fact that in this limit $a_1\to 0$.

Bound states are located at

$$E_n=-\gamma^4\mu_n^2\ , \tag{108}$$

where n varies from 0 to $n_{\max}$ defined below and

$$\mu_n = \frac{1}{\gamma^2}\left[-\Big(2n+\lambda+\frac{1}{2}\Big) + \left[\Big(2n+\lambda+\frac{1}{2}\Big)^2(1-\gamma^2) + \gamma^2\Big(s+\frac{1}{2}\Big)^2\right]^{\frac{1}{2}}\right] . \tag{109}$$

All the terms in (105) are finite at the origin, with the exception of the last one, which shows r^{-2}-like singularity there, and can be considered either as an approximation of the centrifugal term with $l = \lambda - 1$ (λ integer), or as a part of a singular potential with arbitrary $l \neq \lambda - 1$. Setting $\lambda = 1$ we get the "simple" Ginocchio potential [13] defined on the line, discussed also in subsection 3.1.2. In what follows we assume that $\lambda \geq 1$ holds.

The bound-state wavefunctions are expressed in terms of Jacobi polynomials

$$\psi_0^{(n)}(r) = \mathcal{N}_n(\gamma^2 + \sinh^2 u)^{\frac{1}{4}}(\sinh u)^{\lambda}(\cosh u)^{-\mu_n-\lambda-\frac{1}{2}} P_n^{(\mu_n,\lambda-\frac{1}{2})}(2\tanh^2 u - 1) \tag{110}$$

which reduce to Gegenbauer polynomials for $\lambda = 1$. The normalization is given by

$$\mathcal{N}_n = \left[\frac{2\gamma^2 n!\,\Gamma(\mu_n+\lambda+n+\frac{1}{2})\mu_n(\mu_n+\lambda+2n+\frac{1}{2})}{\Gamma(\mu_n+n+1)\Gamma(\lambda+n+\frac{1}{2})(\mu_n\gamma^2+\lambda+2n+\frac{1}{2})}\right]^{\frac{1}{2}} . \tag{111}$$

Considering that the $r \to \infty$ asymptotical limit corresponds to $u \to \infty$ (see (106)), the wavefunctions become zero asymptotically if $\mu_n > 0$ holds. Applying this condition to (109) we find that the number of bound states is set by $n_{\max} < \frac{1}{2}(s-\lambda)$.

Later on we shall use the Jost solutions with potential (105) satisfying [131]

$$f_0^{\text{Jost}}(k,r) \to_{r\to\infty} \mathrm{i}^{\lambda-1}\exp(\mathrm{i}kr) . \tag{112}$$

They can be expressed in terms of the two linearly independent solutions for arbitrary energy $E = k^2$ and can be written as

$$f_0^{\text{Jost}}(k,r) = \exp(\mathrm{i}kr_1)(\gamma^2+\sinh^2 u)^{\frac{1}{4}}(-\mathrm{i}\sinh u)^{1-\lambda}(\cosh u)^{-\mu(k)+\lambda-\frac{3}{2}}$$

$$\times F\Big(\frac{1}{2}\Big(\mu(k)-\lambda+\sigma(k)+2\Big), \frac{1}{2}\Big(\mu(k)-\lambda-\sigma(k)+1\Big); \mu(k)+1; \frac{1}{\cosh^2 u}\Big) , \tag{113}$$

where

$$\sigma(k) = -\frac{1}{2} + \Big[\mu^2(k)(1-\gamma^2) + (s+\frac{1}{2})^2\Big]^{\frac{1}{2}} , \tag{114}$$

$$\mu(k) = -\frac{\mathrm{i}k}{\gamma^2} \tag{115}$$

and

$$r_1 = \frac{1}{\gamma^2}\Big[(\gamma^2-1)^{\frac{1}{2}}\tan^{-1}(\gamma^2-1)^{\frac{1}{2}} - \ln(\frac{\gamma}{2})\Big] \tag{116}$$

as in [14]. The Jost solutions allow expressing the Jost function as

$$\begin{aligned} F_0(k) &= \Big(-\frac{k}{2}\Big)^{\lambda-1}\frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda-\frac{1}{2})}\lim_{r\to 0}\Big(r^{\lambda-1}f_0^{\rm Jost}(k,r)\Big) \\ &= \Big(-\frac{\mathrm{i}k}{2}\Big)^{\lambda-1}\frac{\pi^{\frac{1}{2}}\gamma^{-\lambda+\frac{3}{2}}\exp(\mathrm{i}kr_1)\Gamma\Big(1+\mu(k)\Big)}{\Gamma\Big(\frac{1}{2}(\mu(k)+\lambda-\sigma(k))\Big)\Gamma\Big(\frac{1}{2}(\mu(k)+\lambda+\sigma(k)+1)\Big)}. \end{aligned} \tag{117}$$

This provides the S-matrix as

$$S_0(k) = \exp(2\mathrm{i}\delta_0(k)) = (-1)^{\lambda-1}\frac{F_0(-k)}{F_0(k)}\ , \tag{118}$$

which, together with the substitutions discussed previously, gives the result of [14], up to a $(-1)^l$ phase. This difference is due to the fact that, for the definition of this S-matrix, we consider here that the Ginocchio potential is a singular potential in the $l=0$ partial wave, rather than a regular potential in the $l=\lambda-1$ partial wave. This is quite natural since we allow here λ to be non integer. This convention also explains the unusual factors of (112), of the first line of (117), and of the definition of the S-matrix in terms of the Jost function in (118), as explained in [131]. As can be verified by equation (118), the S-matrix tends to 1 for $k\to 0$, and to $\exp[\mathrm{i}\pi(1-\lambda)]$ at infinity. This is in accordance with the Levinson theorem, generalized for singular potentials [41],

$$\delta_0(0) - \delta_0(\infty) = \Big(n_{\max}+1+\frac{\lambda-1}{2}\Big)\pi. \tag{119}$$

3.1.3 A Natanzon potential from a point canonical transformation

We start with presenting the potentials introduced by Dutt et al. [23] as conditionally exactly solvable (CES) models. The two potentials defined on the full axis $x\in(-\infty,\infty)$ can be written in a common form as

$$V^{(g_0,g_1,g_2,g_3)}(x) = \frac{g_0}{e^x\,z(x)} + \frac{g_1}{z(x)} + \frac{g_2}{z^2(x)} + \frac{g_3}{z^4(x)}\ , \tag{120}$$

with $z(x)=(1+e^{-2x})^{1/2}\in(1,\infty)$. The explicit form of these potentials [23] is

$$V_1^{(DKV)}(x) = V^{(0,-B,A,-3/4)}(x), \qquad V_2^{(DKV)}(x) = V^{(-B,0,A,-3/4)}(x)\ . \tag{121}$$

These potentials depend on two parameters (A and B) which define the potential shape. The coupling constant of the third potential term has to be fixed to a constant value ($-3/4$) in order to obtain exact solution of these models. This is why the authors of [23] identified these potentials as conditionally exactly solvable (CES) ones.

One can easily demonstrate that the two potentials, in fact, are equivalent [P11] in the sense that

$$V^{(0,-B,A,-3/4)}(x) = V^{(-D,0,C,-3/4)}(-x) + \varepsilon \ , \tag{122}$$

where

$$\varepsilon = -A + 3/4, \qquad C = -A + 3/2, \qquad D = B \ . \tag{123}$$

Thus, in what follows it is sufficient to deal with only one of the potentials, so we pick $V_1^{(DKV)}(x)$ for our analysis [P11].

In [23] potentials (121) were introduced using the point canonical transformation method [117], by which a Schrödinger-type differential equation can be transformed into another equation of this type, applying an invertible parametrization $r = r(x)$. With this change of variables, dating back to Liouville [118] a given asymptotically free equation

$$\left[-\frac{d^2}{dr^2} + U(r) \right] \chi(r) = -\kappa^2 \, \chi(r) \tag{124}$$

can be transformed into an apparently different bound state problem

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \psi(x) = -k^2 \, \psi(x). \tag{125}$$

After we denote the derivative by a prime ($x'(r)$ etc.), an elementary correspondence between the potentials and/or energies is obtained,

$$U(r) + \kappa^2 = [x'(r)]^2 \left(V(x(r)) + k^2 \right) + \left(\frac{3}{4} \frac{x''(r)}{x'(r)} \right)^2 - \frac{1}{2} \frac{x'''(r)}{x'(r)} \ . \tag{126}$$

Obviously, the "old" energy eigenvalues are related to the parameters of the "new" potential, and vice versa. The formal definition of the new wavefunctions is also virtually trivial,

$$\psi(x) = [x'(r(x))]^{1/2} \, \chi(r(x)) \ . \tag{127}$$

In any situation of practical interest one may just pick a suitable exactly solvable (ES) problem (124) and derive quickly its partner (125). Setting out

from two shape-invariant [18] ES potentials defined on the positive half axis, Dutt et al. [23] used the variable transformation $x = \ln(\sinh r)$ to obtain potentials (121). The particular initial potentials and their energies were

$$U_1(r) = -2b\frac{\cosh\, r}{\sinh\, r} + a(a-1)\,\frac{1}{\sinh^2 r}, \quad \kappa^2 = \kappa_m^2 = (a+n)^2 + b^2/(a+n)^2 \quad (128)$$

(with $b > (a + n_{\max})^2$) and

$$U_2(r) = -(2a+1)b\,\frac{\cosh\, r}{\sinh^2 r} + \left[a(a+1) + b^2\right]\,\frac{1}{\sinh^2 r}, \quad \kappa^2 = \kappa_n^2 = (a-n)^2 \quad (129)$$

(with $b > a > n_{\max}$). This can be recognized as the Eckart potential in table 1.

Recalling the bound-state wavefunctions of potentials $U_j(r)$, the solutions to potentials $V_j^{(DKV)}(x)$ in (121) readily follow from (127). Without the loss of generality we can consider the $j = 1$ case and recall the solutions of $U_1(r)$ (see e.g. [10, 3]) in terms of Jacobi polynomials,

$$\chi(z) = (z-1)^{-\frac{1}{2}(a+n-s)}(z+1)^{-\frac{1}{2}(a+n+s)}P_n^{(-a-n+s,-a-n-s)}(z), \quad s = b/(a+n) \quad (130)$$

with $z = z(r) = \coth r$. Using this function in (127), substituting it into the Schrödinger equation and matching parameters a and b with A and B of $V_1^{(DKV)}(x)$ in (121), we find $B = 2b$ and

$$A = n^2 + 1/2 + (2n+1)\,a + b^2/(a+n)^2. \quad (131)$$

This equation will ultimately determine the energy eigenvalues through a cubic equation of quantum number n, as described also in [23].

Let us now continue with the analysis of the energy eigenvalues based on the formula (131). The key element of our approach is the strict observation of the constraints imposed on the parameters by the boundary conditions of the wavefunctions. By this we mean both the solutions of the "old" potential $U_1(r)$ (128) and those of the "new" one $V_1^{(DKV)}(x)$ (121).

The appropriate physical boundary condition for (130) near the threshold $r \to 0$ implies that we have to choose $a > 1/2$. Then, after the transition from r to x we get the wavefunctions still safely normalizable near the left infinity $x \to -\infty$. Similarly, our explicit wavefunctions remain asymptotically normalizable near the right infinities $r \to \infty$ and $x \to +\infty$ if and only if we have $a + n < b/(a+n)$. This means that the eligible quantum numbers $n = 0, 1, \ldots, M$ have to be such that $0 \le M < b^{1/2} - a$, i.e.,

$$(n + 1/2)^2 < (a+n)^2 < b. \quad (132)$$

Without presenting the details, we just state the main result of the analysis given in [P11] regarding the choice of the physical solution of the cubic algebraic equation (131): the general rule is that always the middle root is the physical one.

Let us now turn to the interpretation of the potential $V_1^{(DKV)}(x)$ in (122). Obviously, the transformation employed in [23] (i.e. the point canonical transformation [117] or the Liouvillean method [118]) is a special case of the transformation method [9, 10] presented in subsection 2.1. Taking

$$Q(z) = 0 \ , \qquad\qquad R(z) = -\kappa^2 - U(z) \ , \tag{133}$$

Equation (7) reduces to the inverted version of (131) (with r and $-k^2$ there replaced with z and E here). Similarly, (8) also reduces to the equivalent of (127), where $\chi(r)$ is playing the role of $F(z)$. From this it is clear that in contrast with the claim of the authors of [23], the potentials in (122) should be referred to as Natanzon-class potentials, rather then CES ones.

The approaches applied in [9, 10] (reviewed in subsection 2.1) and in the point canonical transformation [117] emphasize somewhat different strategies of deriving solvable potentials within the Natanzon potential class [8]. In [9, 10] the main point is to identify some term on the right-hand side of (7), to account for the constant (i.e. the energy) term on the left-hand side. With this, a differential equation of the type (9) is obtained, and this determines the function $z(x)$ describing the variable transformation. In some cases the $z(x)$ function could not be determined explicitly from (9), only the inverse $x(z)$ function, therefore a number of solvable models obtained this way turned out to be "implicit" potentials. On the other hand, following the point canonical transformation method [117], the $z(x)$ function is always available in an explicit form, however, it is not guaranteed that any $z(x)$ function would lead to a Schrödinger-like equation in which all the n-dependence can be absorbed into the constant (energy) term. Equation (131) might turn out to have Sturm–Liouvillean form, where n typically appears in coordinate-dependent terms. Simply stated, the approach of [10] focuses on having the energy in a simple form, even on the expense of leaving the solutions in a complicated (implicit) form, while in the point canonical transformation the preference is having the solutions in an explicit form, rather than getting the energy expression in a simple way. We stress that despite this difference, the two approaches are interrelated, and are special cases of deriving Natanzon-class potentials.

Let us now see how potential $V_1^{(DKV)}(x)$ in (121) can be obtained from the method described in subsection 2.1. The choice of

$$z^2 (z')^2 (1 - z^2)^{-2} = C \tag{134}$$

in (89) was not discussed in detail in [10], only the generic form of the solution was mentioned. However, it turns out, that the function $z(x) = [1 + \exp(2C^{1/2}x + D)]^{1/2}$ satisfies (134), and it leads to the same variable transformation as that discussed in [23], if the $C^{1/2} = -1$ and $D = 0$ choice is made. The actual form of (7) is now

$$E_n - V(x) = -\left(n + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{1}{2}(\beta-\alpha)(\beta+\alpha)z^{-1}(x) + \frac{3}{4}z^{-4}(x) + \left[\left(n + \frac{\alpha+\beta+1}{2}\right)^2 - \left(\frac{\alpha+\beta}{2}\right)^2 - \frac{3}{4} - \frac{1}{4}(\beta-\alpha)^2\right] z^{-2}(x) \ . \quad (135)$$

This leads to a solvable potential if the n-dependence can be canceled in the coordinate-dependent (i.e. potential) terms by a suitable change of the parameters. Comparing (135) with (121) we get

$$A = -\left[\left(n + \frac{\alpha+\beta+1}{2}\right)^2 - \left(\frac{\alpha+\beta}{2}\right)^2 - \frac{3}{4} - \frac{1}{4}(\beta-\alpha)^2\right] \ , \quad (136)$$

$$B = \frac{1}{2}(\beta-\alpha)(\beta+\alpha) \ , \quad (137)$$

and

$$E_n = -\left(n + \frac{\alpha+\beta+1}{2}\right)^2 \ . \quad (138)$$

Obviously, α and β depend on n and also on the potential parameters A and B. Substituting (138) in (136) and combining it with (137) we arrive at (131), the equation defining the energy eigenvalues.

The bound-state wavefunctions are found to be

$$\psi(x) \sim z^{1/2}(x)(z(x)+1)^{\beta_n/2}(z(x)-1)^{\alpha_n/2}P_n^{(\alpha_n,\beta_n)}(z(x)) \ , \quad (139)$$

which (apart from some misprints), corresponds to equations (15), (16) and (18) in [23], if we substitute $\alpha_n = B/(2c) - c$ and $\beta_n = -B/(2c) - c$.

Finally, it is worthwhile to analyze this potential in terms of the formalism of Natanzon potentials, as discussed in subsection 2.1. It is particularly instructive to examine the role of the 3+3 parameters appearing in the Natanzon potentials, as it is related to the concept of conditionally exact solvability. For the most commonly occuring potentials (like the shape-invariant ones [18]), the three parameters determining the $z(x)$ function via (13) and (12), usually

only one appears, and even that one is a trivial scaling parameter of the coordinate and/or the energy scale. (Trivial coordinate shifts can also appear through them.) Usually they play a non-trivial role only in the case of some "implicit" potentials [13, 62, 119, 16, P11]. In the present case the differential equation (134) corresponds to taking $a_1 = 4c_1 = 4c_0 = 4/C$, and on this basis we can identofy the $V_i^{(DKV)}(x)$ potentials with those dicussed in [17].

The other three parameters appearing in (11) set the potential shape, and determine the relative strength of the individual potential terms. In most potentials only one or two of these parameters appear. The two parameters appearing in potential (121), A and B are of this type. (There could be one more parameter setting the length scale, but it is set to 1 in this case.) Obviously, when there are three potential terms, as in (121), and only two parameters, then the relative strength of the three potential terms cannot be arbitrary, and has to be constrained. This is why the third term of (121) is a numerical constant, i.e. $-3/4$. It is the presence of this numerical constant which earned potentials in [22, 23] the name "conditionally exactly solvable". In fact, based on the structure of their eigenfunctions, the potentials appearing in [23] are of the Natanzon type [8], while those in [22] belong to the Natanzon confluent class [12]. There are, however, further considerations regarding normalizability and regularity, which might impose restrictions on the solvability of certain potentials. Not surprisingly, these may play a more important role in the case of the less "trivial" potentials [120].

3.2 Supersymmetric quantum mechanics

This subsection is divided into three parts containing results from three fields: single and iterated SUSYQM transformations, as well as the generalization of the factorization technique to spin degrees of freedom.

3.2.1 Single supersymmetric transformations

Here we discuss conditionally exactly solvable (CES) potentials generated from supersymmetry as the supersymmetric partners of some simple potentials [121, 122]. The CES nature of these potentials hinges on the question whether the parameters of their partners can be chosen in such a way that they can be reduced to some simple potential with known solutions and energy eigenvalues. According to the techniques of supersymmetric quantum mechanics, the CES potentials constructed in this way are then essentially isospectral with their partners, i.e. the two spectra are identical or differ only in their ground state.

The bound-state solutions of CES potentials are obtained from those of their simple (Natanzon-type) partner potentials by acting on these latter ones with linear differential operators. In [121, 122] some CES potentials have been constructed by SUSYQM. Here we show that this procedure can be made more systematic by making use of various types of SUSYQM transformations [P7]. The rather general nature of this treatment allows the recovery of known results and also the derivation of new CES potentials in the same framework. Our examples concern CES potentials related to the harmonic oscillator potential in three or one dimension (the standard examples of [121, 122]), but the formalism is equally applicable to other types of potentials as well.

Let us start with presenting the conventional SUSYQM approach to CES potentials [121, 122]. Let us assume that there is a pair of SUSYQM partner potentials $V_{\pm}^{(0)}(r)$, which can be constructed from a superpotential $W_0(r)$ in the usual way:

$$V_{\pm}^{(0)}(r) = W_0^2(r) \pm W_0'(r) \ . \tag{140}$$

Consider now a superpotential of the form

$$W(r) = W_0(r) + w(r) \ . \tag{141}$$

The partner potentials generated from $W(r)$ are then

$$V_+(r) = V_+^{(0)}(r) + 2W_0(r)w(r) + w^2(r) + w'(r) \tag{142}$$

$$V_-(r) = V_-^{(0)}(r) + 2W_0(r)w(r) + w^2(r) - w'(r) \ . \tag{143}$$

Let us now insist on that one of these potentials, say $V_+(r)$ is related to some known potential up to an energy shift. In the simplest case this could be $V_+^{(0)}(r)$ in (140):

$$V_+(r) = V_+^{(0)}(r) + \Delta \ . \tag{144}$$

Combined with (142), this requirement immediately introduces a Riccati-type differential equation for $w(r)$:

$$w^2(r) + w'(r) + 2W_0(r)w(r) = \Delta \ . \tag{145}$$

If this equation is solved, then a pair of SUSYQM potentials is obtained, from which one of the partner potentials, $V_+(r)$, corresponds to a known potential (up to an energy shift). Therefore, both the spectrum and the wavefunctions of the partner potential $V_-(r)$ can be obtained in the usual way.

In the examples in [121] $V_+^{(0)}(r)$ was the harmonic oscillator potential in 1 and 3 dimensions, with $W_0(r)$ being the corresponding superpotential. In both cases the structure of $w(r)$ was of the type

$$w(r) = \sum_{i=1}^{N} \frac{2g_i r}{1 + g_i r^2} \ . \tag{146}$$

In the practical examples N=1 was used.

Let us now present a more general alternative SUSYQM construction of CES potentials. As discussed in subsection 2.2, a potential $V_1(r)$ isospectral with a known potential $V_0(r)$ can be constructed by (44), where $\phi(r)$ is a solution of the Schrödinger equation with potential $V_0(r)$ and ϵ is the factorization energy. Depending on the value of ϵ and the boundary conditions of the solution $\phi(r)$, $V_1(r)$ in (44) will have various properties. As discussed in subsection 2.2 (see table 2), for a radial problem (in three-dimensions) four types of transformations are possible. These are related four different types of *nodeless* solutions $\phi(r)$ of the Schrödinger equation. The nodelessness of $\phi(r)$ guarantees that the resulting potential $V_1(r)$ does not have singularities for finite values of r (besides the origin), and this can be achieved whenever the factorization energy ϵ is below the ground-state energy of $V_0(r)$ [36].

Let us consider the radial harmonic oscillator as an example and solve the Schrödinger equation for $\phi(r)$ with

$$\begin{aligned} V_0(r) = V_+^{(0)}(r) &= W_0^2(r) + W_0'(r) \\ &= r^2 + \frac{\gamma(\gamma+1)}{r^2} + 2\gamma + 3 \ . \end{aligned} \tag{147}$$

Here the superpotential is $W_0(r) = r + (\gamma+1)r^{-1}$, and the bound states of $V_0(r)$ are found at $E_n = 4n + 4\gamma + 6$. The solution $\phi(r)$ can be searched for in the form

$$\phi(r) \simeq r^A \exp\left(\frac{B}{2} r^2\right) F(a, b; Cr^2) \ , \tag{148}$$

where $F(a,b;z)$ is the confluent hypergeometric function [19]. Straightforward calculation shows that the Schrödinger equation transforms into the confluent hypergeometric equation if the following conditions hold:

$$A(A-1) = \gamma(\gamma+1) \ , \qquad B^2 = 1 \ , \qquad B = -C \ ; \tag{149}$$

$$b = A + \frac{1}{2} \tag{150}$$

$$a = -\frac{\epsilon}{2C} + \frac{\gamma}{2C} + \frac{3}{4C} + \frac{A}{2} + \frac{1}{4} \ . \tag{151}$$

Recalling that besides $F(a,b;z)$, $z^{1-b}F(a-b+1,2-b;z)$ is a linearly independent solution of the same confluent hypergeometric function [19], the general solution $\phi(r)$ has the form

$$\phi(r) \simeq \exp\left(\frac{B}{2}r^2\right)\left[\alpha_1 r^{\gamma+1}F(\frac{\epsilon}{2C}+\frac{\gamma}{2C}+\frac{3}{4C}+\frac{\gamma}{2}+\frac{3}{4},\gamma+\frac{3}{2};Cr^2)\right.$$
$$\left.+\alpha_2 r^{-\gamma}F(\frac{\epsilon}{2C}+\frac{\gamma}{2C}+\frac{3}{4C}-\frac{\gamma}{2}+\frac{1}{4},-\gamma+\frac{1}{2};Cr^2)\right] . \qquad (152)$$

Note that the two terms in (152) are connected by the $\gamma \leftrightarrow 1-\gamma$ transformation, therefore it is enough to consider one of the solutions ($A=\gamma+1$ or $A=-\gamma$) of $A(A-1)=\gamma(\gamma+1)$ in (149). The solutions corresponding to the transformations T_1, T_2, T_3 and T_4 in table 2 can then be identified by imposing the appropriate boundary conditions on $\phi(r)$.

Substituting the $\phi(r)$ function in (44) one obtains an expression for $V_1(r)$ in terms of $\phi'(r)$ and $\phi''(r)$. Ultimately $V_1(r)$ can be expressed in terms of $V_0(r)$, ϵ and ϕ'/ϕ. In this last expression the first-order derivatives of two confluent hypergeometric functions occur, each of which can be expressed in terms of another confluent hypergeometric function [19]. This means that $V_1(r)$ can be expressed in a somewhat complicated, but closed analytic form. A special situation occurs when $a=-N$ or $a-b+1=-M$ holds. In this case one of the confluent hypergeometric functions occuring in (152) reduces to an N-th or M-th order (generalized Laguerre [19]) polynomial of the argument. According to (151), this case corresponds to specific choices of the factorization energy ϵ. Let us now consider the four transformations T_1, T_2, T_3 and T_4 one by one [P7].

We first note that the boundary conditions require $\alpha_2=0$ in the T_1 and T_3 cases, while in the T_2 and T_4 cases both α_1 and α_2 are allowed, and their ratio appears in $V_1(r)$ as a new parameter. However, for simplicity we consider only $\alpha_1=0$ in the T_2 and T_4 cases and also reduce the remaining confluent hypergeometric function to an N'th order polynomial by setting its first parameter to $-N$. With these choices $V_1(r)$ can be written in a compact form:

$$V_1(r)=r^2+\frac{\gamma(\gamma+1)+2A}{r^2}+2\gamma+3-2B-2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln F(-N,A+\frac{1}{2};Cr^2) . \qquad (153)$$

The solutions relevant to the T_1, T_2, T_3 and T_4 cases can then be obtained by substituting $[A,B,C]=[\gamma+1,-1,1]$, $[-\gamma,1,-1]$, $[\gamma+1,1,-1]$ and $[-\gamma,-1,1]$, respectively. In the $N=0$ case the last term in (153) cancels and $V_1(r)$ contains only terms characteristic of the three-dimensional harmonic oscillator

potential. For $N=1$, $F(-1,A+\frac{1}{2};Cr^2)=1+g_1r^2$, with $g_1=-2C/(2A+1)$, which gives rise to two new terms [P7]

$$V_1(r)=r^2+\frac{\gamma(\gamma+1)+2A}{r^2}+2\gamma+3-2B+\frac{8g_1^2r^2}{(1+g_1r^2)^2}-\frac{4g_1}{1+g_1r^2}\ . \quad (154)$$

In the T_1 case $A=\gamma+1$, $B=-1$ and $C=1$ has to be taken. The $a=-N$ condition leads to $\epsilon=4N+4\gamma+6$. This factorization energy corresponds to the bound-states energies of $V_0(r)=V_+^{(0)}(r)$ and $\phi(r)$ simply reproduces the physical wavefunctions. However, only the $n=0$ ground-state wavefunction is nodeless, so only this can lead to singularity-free $V_2(r)$. Therefore the T_1 transformation simply retrieves the classic SUSYQM transformation which eliminates the ground state of $V_0(r)$ and increases the value of γ with one unit.

In the T_3 case the appropriate choice is $A=-\gamma$, $B=1$ and $C=-1$. The $a=-N$ polynomial condition then leads to the specific factorization energies $\epsilon=-2N$, which are always below the ground-state energy of $V_0(r)$, so the nodelessness of $\phi(r)$ is always secured. The $N=0$ choice recovers $V_1(r)$ as another oscillator with the same spectrum as $V_1(r)$: only the value of γ is increased with one unit and the energy is shifted downwards with two units. The $N=1$ case results in the CES potential described in [121] (denoted by $V_-(r)$ there) up to an energy shift. Similar, but more complicated isospectral potentials would arise from choosing $N>1$.

In contrast with the previous two cases, for the T_4 transformation the boundary condition at the origin now allows both the regular and the singular solution in (152). Similarly to the T_3 case, the T_4 one is usually also interpreted as a situation with broken supersymmetry, because the spectra of the partner potentials (and, of course, of the whole family) is identical. Simplifying the problem by considering only $\alpha_1=0$ in (152) we get the resulting potential from (153) with $A=-\gamma$, $B=-1$ and $C=1$. The $a-b+1=-N$ polynomial condition now leads to factorization energies $\epsilon=4N+4$. The $N=0$ choice again results in another harmonic oscillator potential, with γ decreased with one unit and with an energy shift of two units upwards. For $N=1$ a potential similar to that in [121] arises, whenever $\gamma>1/2$ holds. (As we have mentioned already, this latter condition secures that the polynomial $F(-1,-\gamma+\frac{1}{2};r^2)=1+2r^2/(2\gamma-1)$ remains nodeless, and there will be no singularities in the $V_1(r)$. In fact, the $4\gamma+6=E_0>\epsilon=4N+4$ condition also leads to $\gamma>1/2$ for $N=1$.) Similarly to the T_3 case, further potentials isospectral with a harmonic oscillator can be constructed by choosing $N>1$, but the nodelessness of $\phi(r)$ has to be checked in each case.

In the T_2 case the $A=-\gamma$, $B=-1$ and $C=1$ choice has to be made,

and the situation is the same as in the T_4 case: both the regular and the singular solutions are allowed by the boundary condition at the origin. This means, that we again have a whole family of potentials $V_2(r)$, which have the same spectrum and differ only in their shape. As before, we again restrict to $\alpha_1 = 0$ and consider the polynomial condition $a - b + 1 = -N$, which leads to $\epsilon = -4N + 4\gamma + 2$. For $N = 0$ $V_2(r)$ is an oscillator with γ replaced with $\gamma - 1$ and shifted lower with two units. Clearly, this corresponds to the usual SUSYQM transformation which inserts a new state (at $E = 4\gamma + 2$) below the ground state of $V_0(r)$. For $N = 1$ we find that the potential (154) is nodeless only if $\gamma < 1/2$ holds. For $N > 1$ we have to check the nodelessness of $\phi(r)$ in each case, because it cannot be automatically guaranteed after we restricted the general solution by selecting $\alpha_1 = 0$ in (152).

The relation of the two procedures outlined above can be interpreted in a simple way by noting that the partner potentials are linked by $V_+(r) - V_-(r) = W'(r)$ and $V_0(r) - V_1(r) = (\ln \phi(r))''$. From this

$$W(r) = (\ln \phi(r))' + c \tag{155}$$

follows. Direct integration of (141) and (146) with $W_0(r) = r + (\gamma + 1)r^{-1}$, as in [121] and $c = 0$, indeed, recovers the general solution $\phi(r)$ specific to the T_3 case:

$$\phi(r) \simeq r^{\gamma+1} \exp\left(\frac{r^2}{2}\right) \Pi_{i=0}^{N}(1 + g_i r^2) \ . \tag{156}$$

In addition to the notation of [121], $g_0 = 0$ was also introduced for convenience. This function is also an N'th order polynomial, as expected from (148) for $a = -N$.

Finally, we note that further single SUSYQM transformations are discussed in subsection 3.5, where the interrelation of different symmetry concepts is analyzed.

3.2.2 Combined supersymmetric transformations and phase-equivalent potentials

Analytic exploration of phase-eqivalent complex potentials

It has been known for a long time that the interaction of composite nuclear objects can be described with potentials essentially differing in their shape and depth. In particular, it was known that a "deep" and a "shallow" family of potentials can account for the same phase shifts. This duality of deep and shallow potentials describing the interaction of *composite* nuclear systems (clusters) is

understood qualitatively on the basis of the Pauli principle [123]. According to this, states with low node number in the relative motion are discarded for deep potentials on grounds that they would correspond to states in which the nucleons of the different clusters occupy the same state of the compound nucleus. In shallow potentials, on the other hand, the Pauli principle is taken care of by a repulsive core, which prevents the two clusters from getting too close to each other.

For *real* potentials, the apparent difference between deep and shallow potentials can be exactly eliminated by constructing phase-equivalent potentials [37]. Indeed, supersymmetric transformations [36], which are based on a factorization of the Hamiltonian, allow to construct potentials which provide exactly the same phase shifts as a given potential [37, 124, 44, 43, 45]. Deep potentials can be transformed into equivalent shallow potentials by removing their unphysical bound states. The resulting shallow potentials display a singularity at the origin which is unavoidable according to the generalized Levinson theorem [41]: the variation of the number of bound states is compensated by the singularity in order to keep a constant difference of phase shifts between zero and infinite energies. As a consequence, the resulting shallow potential usually depends on the angular momentum.

However, realistic heavy-ion collisions are not restricted to a single channel. In order to take absorption into account, complex optical potentials need to be used [76]. Is it possible to transform a complex potential with a deep real part into potentials with a shallow real part and to maintain the phase shifts in the process, similarly to real potentialsΓ To answer this question one has to venture into largely unexplored territories. The main difficulty here is finding normalizable solutions of complex potentials. Several questions arise. How many normalizable solutions can one find for a given complex potentialΓ Which types of square-integrable solutions exist in a general caseΓ Can any of them be removed and should it be doneΓ Indeed it is known that the real part of their energy is not necessarily negative [125]. This new type of "bound state" has no clear physical meaning. Do the normalizable solutions present nodes and, if so, does it matterΓ

In [P3] these questions were addressed, and phase-equivalent complex potentials have been constructed numerically. In order to aid the numerical studies, the exactly solvable complex Pöschl–Teller potential was also studied [P3], in the hope that potentials that have similar shape would lead to similar results using numerical techniques.

Before turning to the particular potential itself, it is worthwhile to summarize what one can know about normalizable solutions in a complex potential.

Consider the Schrödinger equation

$$\left(-\frac{d^2}{dr^2} + U(r) + \mathrm{i}W(r)\right)\psi(r) = E\psi(r) \tag{157}$$

with the condition $W(r) < 0$ and assuming that the real part $U(r)$ includes the centrifugal term.

We are interested in normalizable (or square-integrable) solutions of (157) with the boundary condition $\psi(0) = 0$. The corresponding complex eigenvalues E are parametrized as $E = -\kappa^2$ with $\kappa = |\kappa|e^{i\alpha}$, where $-\frac{\pi}{2}\pi < \alpha < \frac{\pi}{2}\pi$, so that $\mathrm{Re}\,\kappa > 0$. For a potential decreasing fast enough, the asymptotic behaviour of a normalizable solution is

$$\psi \underset{r\to\infty}{\to} \exp(-\kappa r) \ , \tag{158}$$

corresponding to an exponentially damped oscillation. When $V(r)$ behaves as a Coulomb potential at large distances, this exponential is multiplied by some (possibly complex) power of r.

The normalizable solutions of (157) and their energies E verify some simple general properties. One easily shows [125] that

$$\mathrm{Im}\,E = \frac{\int_0^\infty |\psi|^2 W dr}{\int_0^\infty |\psi|^2 dr} \ . \tag{159}$$

Hence the existence of a minimum $W_{\min}$ for $W(r)$ and the condition $W(r) < 0$ lead to bounds for the imaginary part of the complex energy, $W_{\min} < \mathrm{Im}\,E < 0$. Moreover, the phase α in κ is positive $0 \le \alpha < \frac{1}{2}\pi$. In a similar way, one shows with a partial integration that

$$\mathrm{Re}\,E = \frac{\int_0^\infty |\psi|^2 U dr}{\int_0^\infty |\psi|^2 dr} + \frac{\int_0^\infty |\psi'|^2 dr}{\int_0^\infty |\psi|^2 dr} \ . \tag{160}$$

Hence, the real part of the eigenvalue is bounded from below $U_{\min} < \mathrm{Re}\,E$. However, $\mathrm{Re}\,E$ can be positive because of the second term in (160), even when $U(r)$ is attractive everywhere. A purely imaginary potential can only support eigenvalues with $\mathrm{Re}\,E > 0$.

Further more complicated bounds [P3] can be obtained by replacing r by $e^{-i\theta}r$ in (157), in the spirit of the complex-rotation technique [126].

In what follows we analyze the solutions of the Pöschl–Teller potential as a radial problem [20], also allowing it to the complex domain. This potential

can be interpreted as the special case of several shape-invariant potentials (see table 1). Here we parametrize it as

$$V(r) = -\frac{s(s+1)}{\cosh^2 r} \ . \tag{161}$$

The energies of the bound states are given by

$$E_n = -\kappa_n^2 = -(s-1-2n)^2 \ , \tag{162}$$

where n satisfies $2n+1 < s$. The corresponding wavefunctions are written in terms of Gegenbauer polynomials as

$$\psi^{(n)}(r) \simeq (\cosh r)^{2n+1-s} C_{2n+1}^{(s-\frac{1}{2}-2n)}(\tanh r) \ . \tag{163}$$

These wavefunctions decay exponentially for the allowed values of n.

For positive energies $E = k^2$ $(k > 0)$, the regular wavefunction reads

$$\psi(r) \simeq \sinh r (\cosh r)^{-s} \times F\left(\frac{1}{2}(-s+1-ik), \frac{1}{2}(-s+1+ik); \frac{3}{2}; -\sinh^2 r\right) , \tag{164}$$

where F is the hypergeometric function. From the asymptotic behaviour one deduces the collision matrix

$$S(k) = A\exp(2i\delta) = -\left(\frac{1}{2}\right)^{2ik} \frac{\Gamma(ik)\Gamma\left(\frac{1}{2}(2+s-ik)\right)\Gamma\left(\frac{1}{2}(1-s-ik)\right)}{\Gamma(-ik)\Gamma\left(\frac{1}{2}(2+s+ik)\right)\Gamma\left(\frac{1}{2}(1-s+ik)\right)}, \tag{165}$$

where A and δ are real. Except for a misprint, the same formula can be found in [31].

Now we choose for s the complex value

$$s = \sigma + i\tilde{\sigma} \ . \tag{166}$$

It corresponds to a potential strength

$$-(u+iw) = -[\sigma(\sigma+1) - \tilde{\sigma}^2] - i\tilde{\sigma}(2\sigma+1) \ , \tag{167}$$

where both u and w should be positive in physical applications. We shall assume that σ and $\tilde{\sigma}$ are both positive.

All the formulae (161) to (165) remain valid for a complex s. The condition for having a square-integrable wavefunction $\psi_n(r)$ becomes $\mathrm{Re}\,\kappa_n > 0$, i.e.

$$\mathrm{Re}\,(\sigma - 2n - 1 + i\tilde{\sigma}) > 0 \quad \Rightarrow \quad 2n+1 < \sigma \tag{168}$$

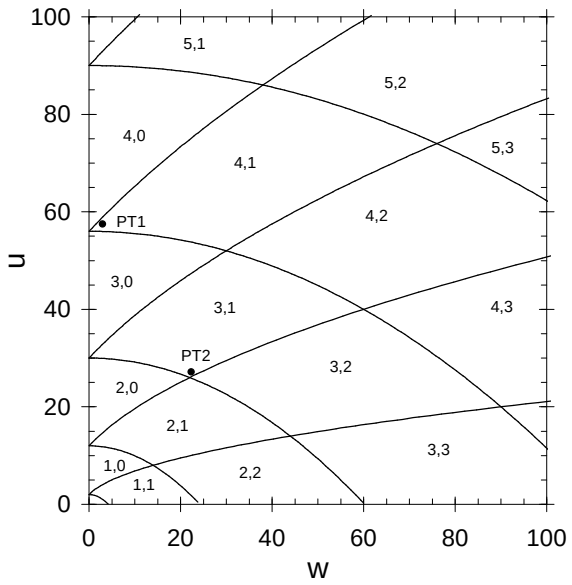


Figure 3: Division of the uw plane according to conditions (168) and (169) for a complex Pöschl-Teller potential with strength $-(u+iw)$: total number of normalizable states (left) and number of eigenvalues with a positive real part (right). The points PT1 and PT2 correspond to the examples discussed in the text.

This is different from the condition $\mathrm{Re}\,(E_n) < 0$ which reads

$$(\sigma - 1 - 2n)^2 - \tilde{\sigma}^2 > 0 \quad \Rightarrow \quad 2n + 1 < \sigma - \tilde{\sigma} \tag{169}$$

The situation is best seen in figure 3 where the uw plane is divided up according to the number of square-integrable states and of states with $\mathrm{Re}\,(E_n) > 0$. This splitting between two kinds of eigenstates naturally appears with the introduction of complex potential parameters. In the real case ($w = 0$) the difference between (168) and (169) disappears. It is clear from the graph that a purely imaginary potential can have normalizable solutions but cannot support states with $\mathrm{Re}\,(E_n) < 0$ (see (160)). A potential with a shape close to that of $V(r)$ in (164), most probably also shows these features. We expect similar properties for normalizable solutions of potentials where the real and imaginary parts have different shapes.

Numerical tests were performed in [P3] for two potentials of the type (164), and the results were compared with the exact ones. The first potential (PT1)

was chosen to have a rather weak imaginary part and four bound states including one with a positive real part, and corresponded to $s = 7.1 + 0.2i$. The results for this potential resembled much to those of bound states in every respect, and the phase-equivalent removal of its state with positive real part of the energy lead to a realistic potential [P3]. As another example we considered a potential (PT2) with a stronger imaginary part, corresponding to $s = 5.1 + 2i$. For this case the numerical methods led to poorer results.

In some cases the phase-equivalent potential obtained after the removal of states showed an oscillatory behaviour, and this finding could be understood in terms of the analytical calculations. In particular, the oscillations could be explained by equation (45): whenever the modulus of the (complex) denominator there is small, oscillations can occur [P3]. These analytical estimates indicated that this unphysical oscillatory behaviour can occur when normalizable solutions with $\mathrm{Re}\,\kappa << \mathrm{Im}\,\kappa$ are removed.

The results with the Pöschl-Teller potentials indicate that the search for normalizable solutions and the construction of phase-equivalent potentials can be performed with high accuracy. In addition to the analysis of the energy eigenvalues and phase shifts, further tests of the numerical methods can be performed by comparing the wavefunctions and the transformed potentials with the corresponding analytical expressions.

Exact analytic formulae for phase-equivalent potentials

The success of analytical methods in aiding numerical analyses also raises the question whether it is possible to find examples where the whole procedure can be performed in an analytical way, i.e. whether there are cases where the resulting potential is obtained in a closed algebraic expression. Efforts in this direction have been limited to some particular examples from the well-known shape-invariant potential class [18]. The ground state of the Coulomb [127, 128], Morse and Hulthén [129] potentials have been removed while keeping the phase shifts unchanged, and somewhat more general transformations have been formulated for the Coulomb [128, 44] potential. Other potentials have also been studied without analyzing the effect of the transformations on their spectra [130]. Apart from their aesthetic value, the importance of fully analytical transformations lies in the fact that exact results can be obtained even in critical conditions when the numerical techniques might not be safely controlled. Handling complex potentials can raise such problems, for example [P3, 6].

The abstract formalism developed for the derivation of phase-equivalent partners of known potentials can be applied to the rather general Natanzon

potential class [8], which contains all the shape invariant potentials [18] as special cases. In order to demonstrate this we derived [P4] potentials which are phase-equivalent with the generalized Ginocchio potential [14], which is probably the most well-known member of the Natanzon potential class. As opposed to other approaches to solvable potentials, where the exact treatment of potentials requires the analytical solution of differential equations, constructing exactly solvable phase-equivalent potentials requires the analytic evaluation of certain definite integrals. Our first results in this field concerned the derivation of phase-equivalent partners of the generalized Pöschl–Teller potential by removing any single bound state, adding a single bound state at specific energies and eliminating the first few bound states [C2]. However, we do not mention these results here separately, because apart from the removal of the first few bound states, they are contained implicitly among the results of our analysis concerning the generalized Ginocchio potential [P4], which contains the generalized Pöschl–Teller potential as a special (shape-invariant) subcase.

Let us consider the generalized Ginocchio potential (105) and assume that we want to eliminate the bound state with quantum number N. Following the notation of (110) the N'th wavefunction can be written in a polynomial form

$$\psi_0^{(N)}(r) = (\gamma^2 + \sinh^2 u)^{\frac{1}{4}} (\sinh u)^{\lambda} (\cosh u)^{-\mu_N - 2N - \lambda - \frac{1}{2}} p_N(\cosh u) \ , \qquad (170)$$

where the coefficients of

$$p_N(\cosh u) = \mathcal{N}_N (\cosh u)^{2N} P_N^{(\mu_n, \lambda - \frac{1}{2})}(2\tanh^2 u - 1) \equiv \sum_{j=0}^{N} c_j^{(N)} (\cosh u)^{2j} \qquad (171)$$

are written as

$$c_j^{(N)} = \mathcal{N}_N (-1)^{N-j} \frac{\Gamma(\mu_N + N + 1)\Gamma(\mu_N + 2N + \lambda + \frac{1}{2} - j)}{j!(N-j)!\Gamma(\mu_N + N + 1 - j)\Gamma(\mu_N + N + \lambda + \frac{1}{2})} \ . \qquad (172)$$

The same formulae hold, of course, for any other bound-state wavefunction, which we label with quantum number n.

In order to derive the new potential and the new bound-state wavefunctions using (45) and (46), the substitutions $\varphi_0(k_0, r) = \psi_0^{(N)}(r)$ and $\varphi_0(k, r) = \psi_0^{(n)}(r)$ have to be made now along with $\beta = -1$ in table 3. The integrals appearing in (45) and (46) can then be expressed in terms of the general formula

$$I_{Nn}(r) = \int_0^r \psi_0^{(N)}(t)\psi_0^{(n)}(t)\mathrm{d}t = \frac{(\sinh u)^{2\lambda+1}}{(\cosh u)^{\mu_N + \mu_n + 2N + 2n + 2\lambda - 1}} G_{Nn}(u) \ , \qquad (173)$$

where $G_{Nn}(u)$ is defined as

$$\begin{aligned} G_{Nn}(u) &= \frac{1}{(2\lambda+1)\gamma^2}\sum_{m=0}^{n+N}\frac{d_m^{(Nn)}(\cosh u)^{2m}}{\mu_N+\mu_n+2N+2n+2\lambda+1-2m} \\ &\times \Big[\frac{(2\lambda+1)(\gamma^2-1)}{\cosh^2 u}+\Big((\mu_N+\mu_n+2N+2n-2m)\gamma^2+2\lambda+1\Big) \\ &\times F\Big(-\frac{1}{2}(\mu_N+\mu_n)-N-n+m+1,1;\lambda+\frac{3}{2};-\sinh^2 u\Big)\Big] \quad . \end{aligned} \tag{174}$$

This expression can be derived using equations 3.194.1 and 2 in [132] after rearranging the summation for the two running indices appearing in the polynomial form of $\psi_0^{(N)}(r)$ and $\psi_0^{(n)}(r)$. This also requires the introduction of the coefficients

$$d_m^{(Nn)} \equiv \sum_{j=\max(0,m-n)}^{\min(m,N)} c_j^{(N)} c_{m-j}^{(n)} \quad . \tag{175}$$

The resulting potential which has bound states at E_n in (108), except for $n=N$ takes the form

$$V_2(r) = V_0(r) + 2\frac{\gamma^2+\sinh^2 u}{\cosh^4 u \sinh^2 u}\Big[\frac{(p_N(\cosh u))^2}{G_{NN}(u)}\Big]^2 - \frac{2\gamma^2(p_N(\cosh u))^2}{G_{NN}(u)}$$

$$\Big[\frac{1}{\gamma^2+\sinh^2 u}-\frac{2\mu_N+4N+2\lambda+1}{\cosh^2 u}+\frac{2\lambda}{\sinh^2 u}+\frac{2}{\cosh u}\frac{p_N'(\cosh u)}{p_N(\cosh u)}\Big] \quad , \tag{176}$$

while the new bound-state wavefunctions are

$$\begin{aligned} \psi_2^{(n)}(r) &= (\gamma^2+\sinh^2 u)^{\frac{1}{4}}(\sinh u)^{\lambda}(\cosh u)^{-\mu_n-2n-\lambda-\frac{1}{2}} \\ &\times \left[p_n(\cosh u) - p_N(\cosh u)\frac{G_{Nn}(u)}{G_{NN}(u)}\right] \quad . \end{aligned} \tag{177}$$

We note that in the $\gamma \to 1$ limit equations (173) to (176) reduce to the corresponding formulae derived for the generalized Pöschl–Teller potential [C2].

Figure 4 shows $V_2(r)$ (as in (176)) obtained by removing the first excited state ($N=1$) of the reference potential. In accordance with the generalized Levinson theorem (119) the $(\lambda-1)\lambda r^{-2}$-type singularity of $V_0(r)$ has changed to $(\lambda+1)(\lambda+2)r^{-2}$ for $V_2(r)$, formally increasing the value of λ with two units. We do not plot here the corresponding wavefunctions, rather refer to [P4]. The Jost function of V_2 is directly related to that of V_0 (see table 3), which is analytically known (e.g. (117)). The S-matrices of V_0 and V_2 are thus identical.

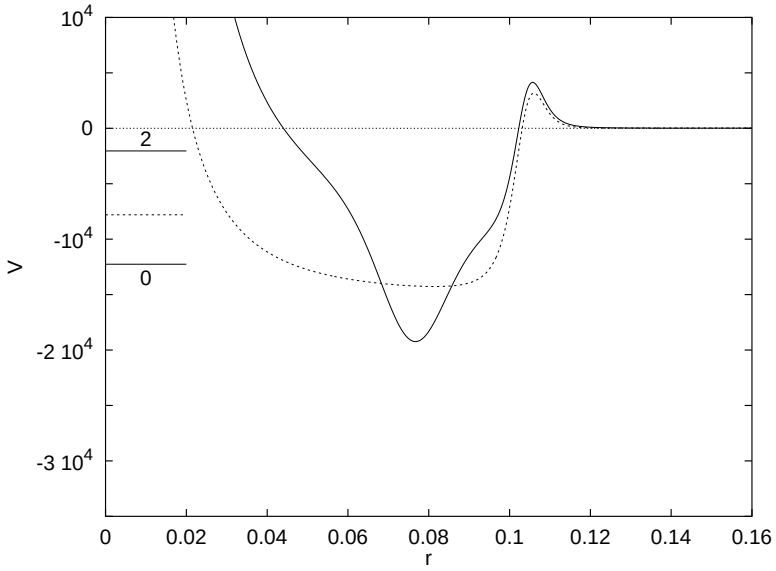


Figure 4: Potential $V_2(r)$ of equation (176), obtained by the removal of the first excited state of the reference potential $V_0(r)$. The reference potential (of (105) with $s = 8$, $\lambda = 3.25$ and $\gamma = 15$) and the eliminated energy level are represented by dashed lines. The $n = 0$ and 2 levels of the reference potential are also shown.

With equation (47), m arbitrary bound states can be removed. For simplicity, we focus on the $m = 2$ case but more general potentials can be obtained for arbitrary m in an obvious way. After removal of the bound states at energies E_{N_1} and E_{N_2}, the potential reads

$$V_4(r) = V_0(r) - 2\frac{\mathrm{d}}{\mathrm{d}r}\,\frac{(\psi_0^{(N_1)})^2 I_{N_2N_2} + (\psi_0^{(N_2)})^2 I_{N_1N_1} - 2\psi_0^{(N_1)}\psi_0^{(N_2)} I_{N_1N_2}}{I_{N_1N_1} I_{N_2N_2} - (I_{N_1N_2})^2} \tag{178}$$

where all functions depend on r which is implied. Note that due to (173) the derivative of $I_{N_iN_j}$ is simply the product of $\psi_0^{(N_i)}$ and $\psi_0^{(N_j)}$. The wavefunctions of the remaining bound states at energies E_n can also be written in terms of the same objects as third-order determinants [43, 45].

The two other options listed in table 3 can also be discussed using similar techniques. These are the phase-equivalent addition of a new bound state at specific energies and changing the potential while leaving the spectrum invariant [P4]. The resulting potentials are expressed in terms of closed formulae similar to (176) with the difference that the parameter β in table 3 remains arbitrary, representing a new potential parameter.

To conclude this subsection we note that the results presented here and in [P4] are the first example for deriving phase-equivalent partners of a potential outside the shape-invariant class using the formalism of SUSYQM. Another novelty was that we also gave closed analytical expressions for the bound-state wavefunctions of the new potential. It is also important to note that the procedure of adding a new bound state to the spectrum requires an r^{-2}-like repulsive singularity of the original potential, therefore it is generalizable only to potentials that have this feature. This forbids a similar treatment of a number of potentials (Morse, Hulthén, Rosen–Morse, etc.).

Similarly to other fully analytical transformations, these results might be helpful in testing numerical methods in situations that might be problematic in terms of numerical techniques. This is the case, for example, for certain types of complex potentials [P3, 130]: the present formulae are applicable to complex Ginocchio potentials without any major modification. The particular case of the Ginocchio potential offers analytical results for a potential with rather flexible shape, which can be considered as a reasonable approximation of realistic potentials used in nuclear physics, for example.

3.2.3 Factorization of spin-dependent Hamiltonians

As it has been discussed in subsection 2.2, isospectral Hamiltonians can be generated by factorizing then in terms of two operators in the following way:

$$H_1 = QR \ , \qquad H_2 = RQ \ . \tag{179}$$

(In this part we use the notation Q and R instead of A and $A^\dagger$ normally used in supersymmetric quantum mechanics.) As discussed previously, besides isospectrality, the Hermiticity of the Hamiltonians H_1 and H_2 can be guaranteed with the additional requirement $Q = R^\dagger$. In the simplest case the one-dimensional (including the radial) Schrödinger equation is factorized, and Q and R are defined as linear differential operators of the type $\pm\frac{\mathrm{d}}{\mathrm{d}x} + W(x)$. The formalism can be developed further by more sophisticated realizations, such as introducing spin variables in Q and R [P6, 133].

Consider the factorization of the Hamiltonians (179) in terms of Q and R defined as

$$Q = \sigma \cdot (\mathbf{p} + \mathbf{a}(\mathbf{r})) + C(r) \ , \qquad R = \sigma \cdot (\mathbf{p} + \mathbf{b}(\mathbf{r})) + D(r) \ , \tag{180}$$

with units $\hbar = 2m = 1$. Assume that C and D are functions of $r = |\mathbf{r}|$, and that

$$\mathbf{a}(\mathbf{r}) = f(r)\ \mathbf{r} \ , \qquad \mathbf{b}(\mathbf{r}) = g(r)\ \mathbf{r} \ . \tag{181}$$

This choice naturally leads to potential problems with spherical symmetry. Substituting (181) into (180) one finds that

$$\begin{aligned} H_1 &= \mathbf{p}^2 + (g+f)\mathbf{r}\cdot\mathbf{p} + \mathrm{i}(f-g)\sigma\cdot\mathbf{L} - \mathrm{i}g'r - 3\mathrm{i}g + gfr^2 + CD \\ &\quad +(C+D)\sigma\cdot\mathbf{p} + (fD+gC-\mathrm{i}\frac{1}{r}D')\sigma\cdot\mathbf{r}\ . \end{aligned} \tag{182}$$

We note that the last two terms of H_1 in (182) have pseudoscalar character. The corresponding formula for H_2 readily follows from (182) by the $f \leftrightarrow g$ and $C \leftrightarrow D$ replacements. We note that (182) can be supplemented with further terms in case we abandon the spherical symmetry by generalizing $\mathbf{a}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ in (180) to

$$\mathbf{a}(\mathbf{r}) = f(r)\ \mathbf{r} + \mathbf{r}\times\mathbf{A}\ , \qquad \mathbf{b}(\mathbf{r}) = g(r)\ \mathbf{r} + \mathbf{r}\times\mathbf{B}\ , \tag{183}$$

where $\mathbf{A}$ and $\mathbf{B}$ are axial vectors [P6].

Table 4 summarizes the conditions under which some of the terms vanish, and also lists the consequences of certain prescribed properties of Q and R. These latter ones include conditions which guarantee the Hermiticity of H_1 and H_2. Table 4 also lists the condition for time reversal invariance requiring that the terms including $\sigma\cdot\mathbf{p}$ and $\sigma\cdot\mathbf{r}$ transform in the same way under time reversal.

In a rather general class of quantum mechanical problems the Hamiltonian is Hermitian, has spherical symmetry, and is free from pseudoscalar and explicitly linear derivative terms. The above conditions are met if $C(r) = D(r) = 0$ and $g(r) = -f(r) = f^*(r)$ hold. The Hamiltonians obtained this way depend on the unspecified function $f(r)$ and describe two non-relativistic problems with spin-orbit interaction:

$$H_1 = \mathbf{p}^2 + 2\mathrm{i}f\sigma\cdot\mathbf{L} + \mathrm{i}f'r + 3\mathrm{i}f - f^2r^2\ , \tag{184}$$

$$H_2 = \mathbf{p}^2 - 2\mathrm{i}f\sigma\cdot\mathbf{L} - \mathrm{i}f'r - 3\mathrm{i}f - f^2r^2\ . \tag{185}$$

The wavefunctions can conveniently be separated into spin functions, spherical harmonics and radial functions [P6].

The construction outlined above can equally be applied to analytically solvable problems and those admitting only numerical solutions. In order to illustrate the procedure, here we consider a problem of the former kind. Substituting $f = \mathrm{i}cr^{-1}$ in (184) and (185) one obtains Hamiltonians in which the spin-orbit interaction appears in a Coulomb-like term:

$$H_1 = \mathbf{p}^2 + c^2 - 2\frac{c}{r}(\sigma\cdot\mathbf{L}+1)\ , \quad H_2 = \mathbf{p}^2 + c^2 + 2\frac{c}{r}(\sigma\cdot\mathbf{L}+1)\ . \tag{186}$$

Table 4: Conditions guaranteeing certain properties of operators Q and R as defined by (180) and (181) and those of Hamiltonians $H_1 = QR$ and $H_2 = RQ$.

Prescription		Conditions	
Properties of Q and R	$Q^\dagger = Q$	$f^*(r) = f(r)$	$C^*(r) = C(r)$
	$R^\dagger = R$	$g^*(r) = g(r)$	$D^*(r) = D(r)$
	$R^\dagger = Q$	$f^*(r) = g(r)$	$C^*(r) = D(r)$
	$R = Q$	$f(r) = g(r)$	$C(r) = D(r)$
	Time reversal invariance	$f^*(r) = -f(r)$	$g^*(r) = -g(r)$
Properties of H_1 and H_2	no $\sigma \cdot \mathbf{p}$ term		$C(r) = -D(r)$
	no $\sigma \cdot \mathbf{r}$ term	$f(r)D(r) + g(r)C(r)$ $-\mathrm{i}r^{-1}D'(r) = 0$	$C'(r) = D'(r)$
	no $\sigma \cdot \mathbf{p}$ and $\sigma \cdot \mathbf{r}$ term	either $g(r) = f(r)$ and	$C(r) = -D(r)$ $= const. \neq 0$
		or	$C(r) = D(r) = 0$
	no $\mathbf{r} \cdot \mathbf{p}$ term	$g(r) = -f(r)$	

Evidently, bound states can appear only when the coefficient of the r^{-1} type term is negative. Without the loss of generality we can assume that $c > 0$ holds: $c \to -c$ merely interchanges H_1 and H_2. Then the sign of the Coulomb term is determined by $\langle \sigma \cdot \mathbf{L} + 1 \rangle$, which is $l + 1$ for $j = l + \frac{1}{2}$ and $-l$ for $j = l - \frac{1}{2}$. The resulting spectra are then

$$E_{nl}^{(1+)} = c^2 \left(1 - \frac{(l+1)^2}{(n+l+1)^2} \right), \quad E_{nl}^{(2-)} = c^2 \left(1 - \frac{l^2}{(n+l+1)^2} \right) , \qquad (187)$$

where the superscripts '+' and '−' stand for states with $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$, respectively. There are no bound states for $E^{(1-)}$ and $E^{(2+)}$.

The above energy eigenvalues can also be obtained from the matrix elements

$$\langle n'(l-1, \frac{1}{2}) jm | Q | n(l, \frac{1}{2}) jm \rangle = -\mathrm{i}\delta_{n',n+1} c \frac{[(n+1)(n+2l+1)]^{\frac{1}{2}}}{n+l+1} , \qquad (188)$$

$$\langle n'(l+1,\frac{1}{2})jm|R|n(l,\frac{1}{2})jm\rangle = \mathrm{i}\delta_{n',n-1}c\frac{[n(n+2l+2)]^{\frac{1}{2}}}{n+l+1} \ . \tag{189}$$

Similarly to the conventional Coulomb problem, the energy levels tend to a well defined value in the $n \to \infty$ limit. This value is not zero, rather it is c^2, due to the different choice of the energy scale. $E = 0$ corresponds now to the ground state of H_1 for the states with $j = l + \frac{1}{2}$, as it can be seen from (187) with $n = 0$. This applies to any value l, so we have an infinitely degenerate ground state for H_1. The corresponding energy levels are missing from the spectrum of H_2, as it can be seen from (187).

The first few energy levels of H_1 and H_2 are plotted in figure 5. The energy levels exhibit a complex degeneracy pattern: $E_{nl}^{(1+)}$ is the same whenever the ratio $(l+1)/(n+l+1)$ has the same value, which can be realized in an infinite variety of ways. (A special case of this is the degeneracy of the ground state with $n = 0$.) Similarly, $E_{nl}^{(2-)}$ has the same value if $l/(n+l+1)$ is fixed.

Another example is obtained by substituting $f(r) = \mathrm{i}\omega/2$ in equations (184) and (185)

$$H_1 = \mathbf{p}^2 - \omega\sigma\cdot\mathbf{L} - \frac{3}{2}\omega + \frac{\omega^2}{4}r^2 \ , \tag{190}$$

$$H_2 = \mathbf{p}^2 + \omega\sigma\cdot\mathbf{L} + \frac{3}{2}\omega + \frac{\omega^2}{4}r^2 \ . \tag{191}$$

These equations describe oscillators, which also experience spin-orbit interaction, the strength of which is correlated with the oscillator constant. With this choice and $j = l \pm \frac{1}{2}$ (190) and (191) lead to radial Schrödinger equations, from which the energy eigenvalues can immediately be determined:

$$E_{nl}^{(1+)} = 2\omega n \ , \qquad E_{nl}^{(1-)} = \omega(2n+2l+1) \ , \tag{192}$$

$$E_{nl}^{(2+)} = \omega(2n+2l+3) \ , \qquad E_{nl}^{(2-)} = 2\omega(n+1) \ . \tag{193}$$

Again we find widespread degeneracy of the states of the two systems. There is an infinite degeneracy for the $E_{n,j}^{(1+)} = E_{n-1,l+1}^{(2-)}$ levels for example, furthermore, we again find that the ground-state energy is infinitely degenerate: $E_{n=0,l}^{(1+)} = 0$ for any l [P6].

We note that similar results have been obtained in a study [134] of the Dirac oscillator [135]. This oscillator problem has also been derived previously in a supersymmetric framework [136] interpreting the "accidental degeneracies" in terms of an su(2)×Osp(2/2) symmetry group.

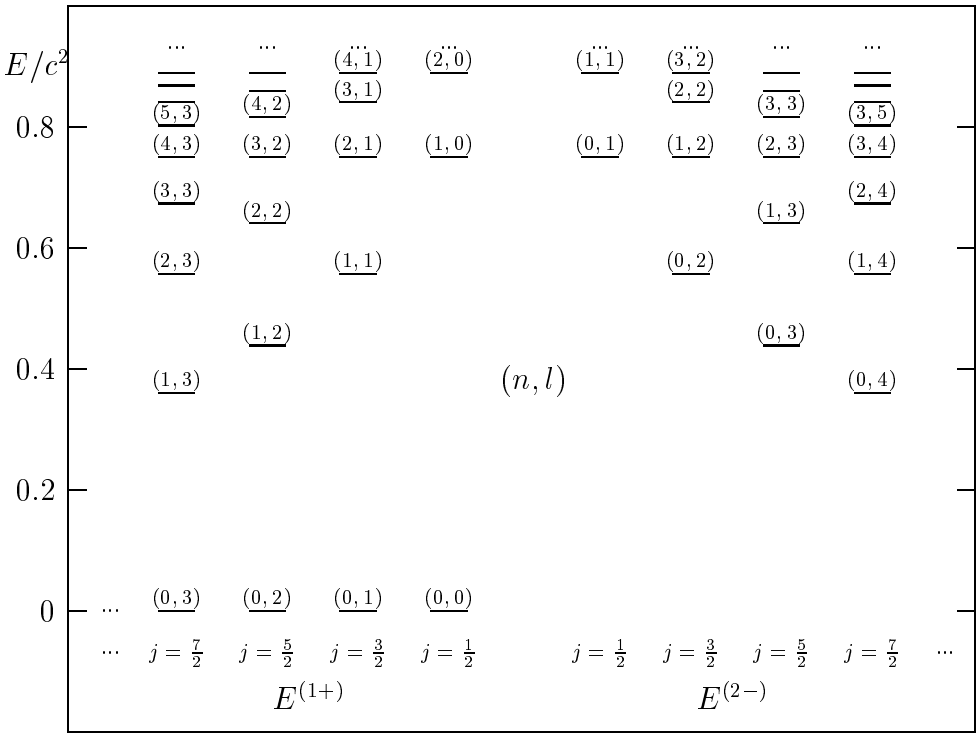


Figure 5: The bound-state energy spectrum of the Coulomb-like problems. The eigenstates of H_1 with $j = l + \frac{1}{2}$ are displayed in the left side of the figure, while those of H_2 with $j = l - \frac{1}{2}$ on the right side. States with the same value of j appear in the same column in both cases. Only states with $E/c^2 \leq \frac{8}{9}$ and $j \leq \frac{7}{2}$ are shown.

With other choices of $f(r)$ discussed in [P6] one arrives at quasi-exactly solvable (QES) problems. The most well-known example for the QES potentials is the sextic oscillator (see e.g. [21])

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \left(\frac{l(l+1)}{r^2} + Ar^2 + Br^4 + Cr^6\right)\psi = E\psi \ , \tag{194}$$

for which the conditions of the quasi-exact solvability are

$$C = a^2 \ , \quad B = 2ab \ , \quad A = b^2 - 2a(2m + l + \frac{5}{2}) \ . \tag{195}$$

Here $a > 0$ and m is a non-negative integer. The solutions ψ can then be written as

$$\psi(r) = Nr^{l+1}\exp\left(-\frac{a^2}{4}r^4 - \frac{b}{2}r^2\right)\Phi_m(r^2) \ , \tag{196}$$

where $\Phi_m(r^2)$ is an m'th order polynomial of r^2 [137]. Hamiltonians of the type appearing in (194) can be obtained from equations (184) and (185) by substituting $f = \mathrm{i}a + \mathrm{i}br^2$ in them, which is clearly a generalization of the oscillator problem discussed before. In this case the quartic and sextic terms in the Hamiltonians $H_i^{(\pm)}$, $i = 1, 2$ will be the same, together with the centrifugal term, while the quadratic and constant terms become l-dependent. However, we find that the condition for quasi-exact solvability (as in (195)) is fulfilled only by $H_1^{(+)}$, and only for $m = 0$, i.e. for the ground state. This ground state is infinitely degenerate [P6], similarly to the Coulomb and harmonic oscillator systems.

Relation to the Dirac equation

In the following example we elucidate the intimate relation between the supersymmetric quantum mechanics (also in the form of the factorization method) and relativistic quantum mechanics [4]. In particular, we present a generalized approach to the Dirac oscillator [135], which also emerged in the present discussion as special case. We present a systematic search for the solutions of the Dirac equation [P2], in a manner somewhat similar to the method applied in subsection 2.1 to the Schrödinger equation. For this we apply an inverse method: we start with a general expression for the minimal and non-minimal couplings in the Dirac equation, and then we reduce this equation to its radial form, in order to study some families of potentials which could be solved exactly or quasi-exactly. ,

We consider the Dirac equation of the form ($c = \hbar = 1$)

$$[\alpha \cdot (\mathbf{p} - \mathrm{i}\beta v(r)\mathbf{r} - u(r)\mathbf{r}) + m\beta - E]\Psi = 0, \tag{197}$$

where $v(r)$ and $u(r)$ are some functions of r and α, β and Ψ are defined as

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} . \tag{198}$$

Squaring the coupled first-order differential equations in the usual way we get for the Ψ_1 component:

$$\Big[\mathbf{p}^2 - 2u(r)\mathbf{r}\cdot\mathbf{p} + (v^2(r) + u^2(r))\mathbf{r}^2 - 2v(r)\sigma\cdot\mathbf{L}$$

$$- r\left(\frac{\mathrm{d}v}{\mathrm{d}r} - \mathrm{i}\frac{\mathrm{d}u}{\mathrm{d}r}\right) - 3(v(r) - \mathrm{i}u(r))\Big]\Psi_1 = (E^2 - m^2)\Psi_1. \tag{199}$$

Separating the radial, angular and spin variables by writing $\Psi_1 = r^{-1} f(r) |(l\frac{1}{2}) j m_j\rangle$ the following radial equation is obtained for $f(r)$:

$$\Big[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2\mathrm{i}ru(r)\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + (rv(r))^2 - \frac{\mathrm{d}}{\mathrm{d}r}(rv(r)) - 2(K+1)v(r) + (ru(r))^2 + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}r}(ru(r)) - \epsilon \Big] f(r) = 0 \ , \tag{200}$$

where

$$K = j(j+1) - l(l+1) - \frac{3}{4} = \begin{cases} l = j - \frac{1}{2} & \text{if } j = l + \frac{1}{2} \\ -l - 1 = -j - \frac{3}{2} & \text{if } j = l - \frac{1}{2} \end{cases} \tag{201}$$

and $\epsilon = E^2 - m^2$.

Considering the following functional form for $f(r)$

$$f(r) = r^{\nu} \exp(-z(r)) \Phi(r) \tag{202}$$

we find after straightforward calculations that this $f(r)$ solves (200) for the special case of $\Phi(r) = const.$ and $\epsilon = 0$ (i.e. $E^2 = m^2$) if $z(r)$ is chosen as

$$z(r) = \int^r r'(-\mathrm{i}u(r') + v(r'))\mathrm{d}r', \tag{203}$$

provided that $\nu(\nu - 1) = l(l+1)$ and $\nu = K + 1$ hold. With $\nu = l + 1$ and $-l$ these are automatically fulfilled for the $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ cases, respectively. So far the functions $v(r)$ and $u(r)$ have not been specified yet.

Further solutions of (200) can be obtained if we choose $\Phi(r)$ as the confluent hypergeometric function $\Phi(r) = F(\sigma, \rho; g(r))$. This substitution yields the following expressions for the unspecified $g(r)$ and $v(r)$ functions:

$$g(r) = a\ r^2 \qquad (a > 0), \qquad\qquad v(r) = a + \frac{b}{r^2} \tag{204}$$

and also $\rho = \nu + \frac{1}{2} + b$ and $\sigma = -\frac{\epsilon}{4a} - \frac{K+1}{2} + \frac{\nu}{2}$ with

$$\nu = \begin{cases} b + \frac{1}{2} + |j - b| & \text{for } \ j = l + \frac{1}{2} \\ b + \frac{1}{2} + |j + 1 + b| & \text{for } \ j = l - \frac{1}{2} \ . \end{cases} \tag{205}$$

Note, however, that we have not obtained any restictions for $u(r)$ yet. The wavefunctions then take the form

$$f^{(+)}(r) = r^{\frac{1}{2}+|j-b|} \exp(-\frac{a}{2}r^2 + \mathrm{i}w(r)) F(-n_r, 2b + 1 + |j - b|; ar^2)$$
$$f^{(-)}(r) = r^{\frac{1}{2}+|j+1+b|} \exp(-\frac{a}{2}r^2 + \mathrm{i}w(r)) F(-n_r, 2b + 1 + |j + 1 + b|; ar^2), \tag{206}$$

where we have used superscripts $(+)$ and $(-)$ to distinguish between the cases for $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$, respectively, and $w(r)$ is defined as $w(r) = \int^r r' u(r') \mathrm{d}r'$. This means that the $u(r)$ function which has not been specified up to this point contributes to a phase factor. The corresponding energy eigenvalues are obtained from

$$\epsilon^{(+)} \equiv (E^{(+)})^2 - m^2 = 2a(2n_r + l - 2j + \frac{1}{2} + b + |j - b|)$$
$$\epsilon^{(-)} \equiv (E^{(-)})^2 - m^2 = 2a(2n_r + l + \frac{1}{2} + b + |j + 1 + b|) \; . \qquad (207)$$

These results [P2] can be interpreted as the generalization of the Dirac oscillator [135], which corresponds to $a = m\omega$, $b = 0$ and $u(r) = 0$. The extension of the Dirac oscillator in [138] is also included in these formulae with $u(r) = 0$, although the energy eigenvalues published in that work differ slightly from those in (207) due to a different parametrization used by the authors. Furthermore, (207) also includes another extension of the Dirac oscillator in [139] as a special case, where a linear potential has been considered in the minimal coupling term in addition to the Dirac oscillator. This situation corresponds to taking $a = m\omega_s$, $b = 0$ and $u(r) = \mathrm{i}m\omega_v$. These authors also noted that the appearance of ω_v does not modify the energy spectrum, and the new term with respect to the Dirac oscillator influences only the form of the wavefunctions. A simple explanation for this result can be given by remembering that $u(r)$ basically represents a phase factor. This is not evident from the formulae presented in [139], nevertheless one should remember that choosing an imaginary, rather than a real $u(r)$ would break the hermiticity of the Hamiltonian in (197).

Finally, we note that the sextic oscillator (194) can also be considered to solve the Dirac equation with a generalization of the Dirac oscillator [P2]. For this $v(r) = c_2 r^2 + c_0$ and $u(r) = 0$ have to be considered, and the solution is trivial for $\Phi(r) = const.$

3.3 Lie-algebraic methods

Lie algebraic methods associated with the relatively simple shape-invariant potentials [18] have already been discussed in much detail (see the review and references in subsection 2.3), so here I discuss algebras associated with certain Natanzon-class potentials [8]. I also discuss the role of various irreducible representations of SU(1,1) when the corresponding su(1,1) algebra plays the role of a spectrum generating algebra.

3.3.1 Realization of the su(1,1) algebra with an extra parameter

Here we first analyze an algebra associated with the Ginocchio potential mentioneded already in subsection 2.1 and 3.1.2, and investigate the role of this algebra in the shape-invariant limit of the Ginocchio potential [C3]. Then we discuss the role of the different irreducible representations of su(1,1) when it plays the role of a spectrum generating algebra assocoated with a singular potential.

The algebraic version of the Schrödinger equation with the Ginocchio potential (102) can be obtained after suitable variable and similarity transformations [C3], which do not change the structure of the original algebra (see subsection 2.3). We then get

$$J_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi} \left(\pm C^{-\frac{1}{2}} (\delta + 1 - z^2(x))^{\frac{1}{2}} \frac{\partial}{\partial x} \mp \frac{\delta}{2} \frac{z(x)}{\delta + 1 - z^2(x)} - z(x) J_z \right), \quad (208)$$

$$|jm\rangle = \mathrm{e}^{\mathrm{i}m\phi} (\delta + 1 - z^2(x))^{\frac{1}{4}} (1 - z^2(x))^{\frac{1}{2}(\alpha - \frac{1}{2})} C_n^{(\alpha)}(z(x)). \quad (209)$$

Direct calculation reveals that the effect of $J_{\pm}$ on the basis states (209) is changing n into $n \pm 1$ while leaving $\alpha = -j = m - n$ unchanged. Since the potential parameters (i.e. Δ and δ in (103)) are interrelated with α and n, we find that the generators ladder between states of potentials with *different* shape in general. We also note that with the $C = \lambda^4(\lambda^2 - 1)^{-1}$ and $\delta = (\lambda^2 - 1)^{-1}$ choice $J_{\pm}$ reproduce the generators presented in [59] for the Ginocchio potential, apart from some minor misprints there.

Let us now consider the special limiting cases [C3]. The $\delta = 0$ choice leads to

$$J_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi} \left(\pm C^{-\frac{1}{2}} (1 - z^2(x))^{\frac{1}{2}} \frac{\partial}{\partial x} - z(x) J_z \right), \quad (210)$$

which ladder between states of the potential $V(x) = C\alpha(\alpha - 1)(1 - z^2(x))^{-1}$ belonging to $E = C(n + \alpha)^2$. Since J_+ and J_- change n with one unit while leaving α unchanged (which, due to $\delta = 0$ is now independent from n) su(1,1) is a spectrum generating algebra here. This is also reflected by the fact that the potential strength depends only on $\alpha(\alpha - 1) = j(j + 1)$, the eigenvalue of the Casimir operator, and the generators ladder between states with different $m = -j + n = \alpha + n$, which now sets the energy.

The $z(x) = \mathrm{i} \sinh x$ and $C = -1$ choice recovers the Pöschl–Teller potential hole with

$$J_{\pm} = -\mathrm{i}\mathrm{e}^{\pm \mathrm{i}\phi} \left(\pm \cosh x \frac{\partial}{\partial x} + \sinh x J_z \right) \quad (211)$$

as generators, found also in [69]. The imaginary factor can be eliminated by a multiplication with i. This amounts to changing the functions $h(x)$, $g(x)$ and $f(x)$ to i$h(x)$, etc., which also turns the algebra into the compact su(2) [69]. This is in line with the fact that the Pöschl–Teller potential has finite number of states. We note that using $z(x) = \sin x$ and $C = 1$ one gets the trigonometric Pöschl–Teller potential, which has infinite number of states and is associated with the non-compact su(1,1) spectrum generating algebra. The singularities of this potential will later on be linked to various irreducible representations of the non-compact $SU(1,1)$ group [C1].

The $\delta \to \infty$, $C\delta^{-1} \to \widetilde{C}$ limit leads to the generators

$$J_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi} \left(\pm \widetilde{C}^{-\frac{1}{2}} \frac{\partial}{\partial x} - z(x) \left(J_z \pm \frac{1}{2} \right) \right). \tag{212}$$

Introducing $\nu(\nu+1) \equiv (\Delta+1)\delta^{-1}$ we find that $V(x) = -\widetilde{C}\nu(\nu+1)(1-z^2(x))$ and $E = -\widetilde{C}(\nu - n)^2 = -\widetilde{C}(\alpha - \frac{1}{2})^2$. The generators (212), again, ladder between states with neighbouring values of n, keeping $\alpha = \nu - n + \frac{1}{2}$ unchanged. This, however, now means that n changes together with ν, thus also with the potential strength. Therefore J_+ and J_- connect states that have the same energy but belong to different potentials, i.e. su(1,1) is a potential algebra here. Contrary to the previous case, E is now related to the eigenvalue of the Casimir operator ($j(j+1)$), while the potential strength is set by the quantum number $m = \nu + \frac{1}{2}$.

The Pöschl–Teller potential hole arises for $z(x) = \tanh x$ and $\widetilde{C} = 1$, which corresponds to the usual Pöschl–Teller limit ($\lambda = 1$) of the Ginocchio potential. $J_{\pm}$ then turn into the standard form of the su(1,1) generators in the potential algebra formalism [61].

In summary, we proved that in two important limiting cases, when the Ginocchio potential reduces to the same types of (Pöschl–Teller-like) potentials, the algebra essentially remains unchanged, but its role becomes different in the two cases: in one limit it is a spectrum generating algebra, while in the second one it appears as a potential algebra. The compactness or non-compactness of the algebra depends on the actual transformation, which is also reflected in the structure of the energy spectrum. For the Pöschl–Teller potential hole the potential algebra was (the non-compact) su(1,1) and the spectrum generating algebra was (the compact) su(2). This situation is reversed if we take the trigonometric version of the Pöschl–Teller potential hole.

We note that the two algebras obtained as the special limits of the same original su(1,1) algebra together are able to connect all the states of a series

of Pöschl–Teller potentials. Such algebras normally appear as subalgebras of some larger algebra.

As discussed in subsection 2.3, various unitary irreducible representations of SU(1,1) are associated with various types of solutions in the potential algebra approach: discrete and continuous unitary irreducible representations are assigned to bound- and scattering-state wavefunctions [61]. This is related to the fact that the Casimir invariant of these problems is related to the Hamiltonian of these potentials by $H = -C_2 - \frac{1}{4}$, so the eigenvalues $j(j+1)$ of C_2 determine whether the energy is in the bound or the scattering domain (i.e. whether it is positive or negative).

This naturally raises the question whether the unitary irreducible representations of SU(1,1) also play such diverse roles when su(1,1) is a spectrum generating algebra. To explore this question, let us consider the differential realization of the su(1,1) algebra (210) with $h(x) = \sin x$, $f(x) = \cos x$, $g(x) = c(x) = 0$. This is a spectrum generating algebra associated with PII type potentials, which one obtains from the eigenvalue equation $(C_2 - j(j+1))|jm\rangle = 0$ of the Casimir operator:

$$\sin^2 x \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{j(j+1)}{\sin^2 x} - m^2 \right) \psi_{jm}(x) = 0 \ . \tag{213}$$

Apart from the $\sin^2 x$ factor this is the Schrödinger equation with the trigonometric Pöschl–Teller potential hole, which is a special case of several PII and PI type shape-invariant potentials listed in table 1. This problem can be looked upon either as a potential restricted to $x = [0, \pi]$, in which case it resembles the infinitely deep rectangular well, or as an element of a periodic potential [C1]. This duality is also connected with the eigenvalue of C_2.

If $j(j+1) > 0$, the potential goes to infinity at the boundaries, thus separating the $x = [0, \pi]$ domain from the others. This is the case when one considers the discrete unitary irreducible representation of $SU(1,1)$ (discrete principal series) D_j^+, with j taking negative integer or half-integer values. The bound states are then associated with m, where its allowed values are [1] $m = -j, -j-1, \ldots$. The operators J_+ and J_- ladder between the bound states and are elements of an su(1,1) spectrum generating algebra [69]. (Due to the $m \to -m$ symmetry of 213 the same problem can also be described by the D_j^- series.)

For the continuous series [1] C_k^0 and $C_k^{1/2}$ $j = -\frac{1}{2} + \mathrm{i}k$ ($k > 0$, real), and m takes on integer or half-integer values. The eigenvalue of C_2 is now $-\frac{1}{4} - k^2$, so that attractive inverse-square-like singularities appear at the boundaries. It is well-known that $-\gamma r^{-2}$-like singularities result in the fall of the particle into

the center of attraction if $\gamma > \frac{1}{4}$ holds [104], and it is obvious that the above case corresponds to such a γ.

There also exist, however, the supplementary series [1, 61] (58) with $-\frac{1}{2} < j < 0$. This unitary irreducible representation of $SU(1,1)$ has not played any role in the potential group approach, where $\langle C_2 \rangle$ was related to the energy eigenvalues, because the discrete and continuous series already accounted for the bound and scattering solutions of the potentials considered there. Here, however, one can identify the supplementary series with "weakly" attractive inverse-square-like potentials, since in this case $-\frac{1}{4} < j(j+1) < 0$ holds, and the complications associated with the "strongly" singular potentials (as above) do not appear. Another peculiar feature of problems with "weakly" attractive inverse-square-like potentials is that both independent solutions are regular at the origin in this case [116].

As we have suggested in [C1], the attractive potential associated with the supplementary series of the $SU(1,1)$ group can also be looked upon as an elementary cell of a periodic potential. In this case modification of the boundary conditions is required, which necessitates the modification of the whole procedure designed initially to study bound-state solutions. We note that in a recent study [140] the supplementary series have indeed been associated with the band spectrum of periodic potentials.

3.3.2 A parameter-free realization of the su(1,1) algebra

As an example for algebras with differential realizations not containing extra parameters, I present an su(1,1) algebra associated with the generalized Coulomb potential. As we have noted previously in subsection 2.3, these realizations typically appear in relation with problems associated with the confluent hypergeometric functions. The speciality of this example is that the differential operators forming the algebra act on the generalized Coulomb–Sturmian basis states [P5], rather than on the physical wavefunctions.

We define the generalized Coulomb–Sturmian equation as a differential equation which has similar structure to the eigenvalue equation (74) with potential (73)

$$\begin{aligned} \hat{X}\phi(\rho,r) \;\equiv\; & \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{3C}{16(h(r)+\theta)^2} + \frac{5C\theta}{16(h(r)+\theta)^3} + \frac{C(\beta-\frac{1}{2})(\beta-\frac{3}{2})}{4h(r)(h(r)+\theta)} \right. \\ & \left. - \left(\frac{\rho^2\theta}{4} + \rho(n+\frac{\beta}{2}) \right) \frac{C}{h(r)+\theta} + \frac{C}{4}\rho^2 \right] \phi(\rho,r) = 0 \ , \end{aligned} \tag{214}$$

and is solved by the generalized Coulomb–Sturmian (GCS) functions [P5]

$$\begin{aligned}\langle r|n\rangle &\equiv \phi_n(\rho,r) \\ &= \left(\frac{\Gamma(n+1)}{\Gamma(n+\beta)}\right)^{1/2}(\rho h(r)+\rho\theta)^{\frac{1}{4}}(\rho h(r))^{\frac{2\beta-1}{4}}\exp(-\frac{\rho}{2}h(r))L_n^{(\beta-1)}(\rho h(r))\ .\end{aligned} \tag{215}$$

Here ρ is a parameter characterizing the generalized Coulomb–Sturmian basis. The GCS functions, being solutions of a Sturm-Liouville problem, have the property of being orthonormal with respect to the weight function $C^{\frac{1}{2}}(h(r)+\theta)^{-1}$. Introducing the notation $\langle r|\widetilde{n}\rangle \equiv \phi_n(\rho,r)C^{\frac{1}{2}}(h(r)+\theta)^{-1}$ the orthogonality and completeness relation of the GCS functions can be expressed as

$$\langle n|\widetilde{n}\rangle = \delta_{nn'} \tag{216}$$

and

$$1=\sum_{n=0}^{\infty}|\widetilde{n}\rangle\langle n| = \sum_{n=0}^{\infty}|n\rangle\langle\widetilde{n}|\ . \tag{217}$$

Straightforward calculation shows that both the overlap of two GCS functions and the $\langle n'|\hat{H}_0|n\rangle$ matrix element can be expressed as a tridiagonal matrix, therefore the matrix elements of the $E-\hat{H}_0$ operator also have this feature [141, P5]:

$$\begin{aligned}\langle n|E-\hat{H}_0|n'\rangle &= \delta_{nn'}\left[\frac{\epsilon}{C^{\frac{1}{2}}\rho}(2n+\beta-\rho\theta)-\frac{C^{\frac{1}{2}}\rho}{4}\left(-\frac{4q}{C\rho}+(2n+\beta)\right)\right] \\ &- \delta_{nn'+1}\left(n(n+\beta-1)\right)^{\frac{1}{2}}\left(\frac{E}{C^{\frac{1}{2}}\rho}+\frac{C^{\frac{1}{2}}\rho}{4}\right) \\ &- \delta_{nn'-1}\left((n+1)(n+\beta)\right)^{\frac{1}{2}}\left(\frac{E}{C^{\frac{1}{2}}\rho}+\frac{C^{\frac{1}{2}}\rho}{4}\right)\ .\end{aligned} \tag{218}$$

This means that similarly to the D-dimensional Coulomb and harmonic oscillator potential, the matrix elements of the Green's operator can be determined by using continued fractions, as described in [142]. The present results, therefore, extend the applicability of this method to a new potential problem. The formulae presented here reduce to those in [142] in the appropriate limits discussed in subsection 3.1.1. The role of the Coulomb–Sturm parameter b used in [142] is now played by $C^{\frac{1}{2}}\rho/2$.

We note that the generalized Coulomb–Sturmian functions can, in principle, be used in calculations for realistic systems such as the $\alpha - \alpha$ cluster configuration of the ^{8}Be nucleus [7].

Having set the basis, an su(1,1) algebra

$$[\hat{J}_1, \hat{J}_2] = -\mathrm{i}\hat{J}_3 \quad [\hat{J}_2, \hat{J}_3] = \mathrm{i}\hat{J}_1 \quad [\hat{J}_3, \hat{J}_1] = \mathrm{i}\hat{J}_2 \tag{219}$$

can be defined in the following fashion:

$$\begin{aligned} \hat{J}_3 &= \frac{h+\theta}{C\rho}\hat{X} + (n + \frac{\beta}{2}) \ , \quad \hat{J}_1 = \hat{J}_3 - \frac{\rho}{2}h \\ \hat{J}_2 &= -\frac{\mathrm{i}}{C^{1/2}}(h(h+\theta))^{1/2}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mathrm{i}\theta}{4(h+\theta)} \ . \end{aligned} \tag{220}$$

As can be seen from equation (214), $\hat{J}_3$ is diagonal in the basis (215) with eigenvalues $m = n + \frac{\beta}{2}$. The elements of this basis can then be associated with the discrete principal series D_j^+ [1] mentioned in (54) and (55), for which the allowed values of m are $m = -j, -j+1, -j+2, \ldots$, with j being negative. It is natural then to identfy j as $j = -\frac{\beta}{2}$. Direct calculations show that the ladder operators connect the neighboring members of this basis:

$$\hat{J}_+\phi_n(\rho, r) \equiv (\hat{J}_1 + \mathrm{i}\hat{J}_2)\phi_n(\rho, r) = [(n+1)(n+\beta)]^{1/2}\phi_{n+1}(\rho, r) \ , \tag{221}$$

$$\hat{J}_-\phi_n(\rho, r) \equiv (\hat{J}_1 - \mathrm{i}\hat{J}_2)\phi_n(\rho, r) = [n(n+\beta-1)]^{1/2}\phi_{n-1}(\rho, r) \ . \tag{222}$$

We find that the eigenvalues of the Casimir invariant

$$\hat{C}_2 = \hat{J}_3^2 - \hat{J}_1^2 - \hat{J}_2^2 \tag{223}$$

are $\frac{\beta}{2}(\frac{\beta}{2} - 1) = j(j+1)$, as expected, and that they set the strength of the fourth term in (214). For $\theta \neq 0$ this is the only singular term and it behaves like γr^{-2} with $\gamma = (\beta - \frac{1}{2})(\beta - \frac{3}{2}) = 4j(j+1) + \frac{3}{4}$. It is interesting to inspect the allowed values of γ for the different unitary irreducible representations of SU(1,1). For the discrete principal series D_j^+ $\beta > 1$ holds, which always secures $-\frac{1}{4} < \gamma$, i.e. the potential has repulsive or "weakly attractive" [116] r^{-2}-type singularity. For the supplementary series [1] $-\frac{1}{2} < j < 0$ holds, which results in $-\frac{1}{4} < \gamma < \frac{3}{4}$. This is exactly the domain where both independent solutions are square integrable at the origin [116]: for $0 < \gamma < \frac{3}{4}$ one of these vanishes at $r = 0$ and the other one is infinite there, while for $-\frac{1}{4} < \gamma < 0$ both solutions vanish at $r = 0$. From (215) it is seen that solutions regular and irregular at the origin correspond to $\beta > \frac{1}{2}$ and $\beta < \frac{1}{2}$. This seems to indicate that one

regular solution (with $\beta > 1$) is associated with D_j^+, while the second square integrable solution, which is either regular or infinite at the origin (depending on β) might be related to the supplementary series, for which $0 < \beta < 1$ holds. For the sake of completeness we note that for the continuous series [1] C_k^0 and $C_k^{1/2}$, $j = -\frac{1}{2} + \mathrm{i}k$ ($k > 0$, real) is valid, which results in the strongly singular $\gamma < -\frac{1}{4}$ case. The solutions then oscillate infinitely near the origin and are unbounded from below [116], which can be interpreted as the falling of the particle into the center of attraction [104]. The situation is similar to that described in [C1] for the $V(x) = \gamma \sin^{-2} x$ potential: the various unitary irreducible representations of the SU(1,1) spectrum generating group there also corresponded to different types of singularities.

We note that we analyzed the singularities of the generalized Coulomb–Sturmian equation (214) and its solutions (215), but similar considerations of the physical potential (73) and its solutions can also be performed taking $D = 3$ and $l = 0$. The algebraic construction, however, does not apply to this latter problem. This is because the bound-state solutions (78) pick up extra n-dependence through ρ_n, which is not accessible for the ladder operators otherwise changing n as in equations (221) and (222).

The present realization of the su(1,1) algebra is a special case of that described in [12] in relation with the Natanzon confluent potentials. Considering the Coulomb and harmonic oscillator limits discussed in subsection 3.1.1 and setting the dimension to $D = 3$, the generators reduce to the forms presented for the two problems separately in [64]. We note that the spectrum generating algebra associated this way with the radial harmonic oscillator problem in three dimensions is different from the one-parameter realization of the su(1,1) algebra discussed in [69, C3], because the ladder operators there are linear differential operators and the Hamiltonian is related to the Casimir invariant, while here the ladder operators are second-order differential operators and the Hamiltonian is essentially a linear function of generator $\hat{J}_3$.

3.4 $\mathcal{PT}$ symmetry of potentials

My results for $\mathcal{PT}$ symmetric potentials are divided here into three main parts. First a class of potentials (mainly shape-invariant ones) are discussed, for which $\mathcal{PT}$ symmetry can be implemented by applying an imaginary coordinate shift. Then the unusual features of $\mathcal{PT}$ symmetric potentials are illustrated with the example of the Scarf II potential, finally, examples are presented for more complicated situations, where the potentials have to be defined along bent contours of the complex x plane in order to make them $\mathcal{PT}$ symmetric.

3.4.1 Potentials generated by an imaginary coordinate shift

Here we apply the procedure outlined in subsection 2.1 to potentials with bound-state solutions containing hypergeometric and confluent hypergeometric functions. It turns out that this method is especially suited to deriving the $\mathcal{PT}$ symmetric versions of shape-invariant potentials, but also those of some more general Natanzon-class potentials. The key element is choosing the otherwise unimportant δ coordinate shift in (10) as an imaginary constant and keeping C real [P9].

First we apply the method to the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ [19]. Modifying somewhat the parametrization used in (135) the actual form of (6) becomes [P9]

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1-z^2(x)}\left(n + \frac{\alpha+\beta}{2}\right)\left(n + \frac{\alpha+\beta}{2} + 1\right)$$
$$+ \frac{(z'(x))^2}{(1-z^2(x))^2}\left[1 - \left(\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2\right] - \frac{2z(x)(z'(x))^2}{(1-z^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \tag{224}$$

Note that in this parametrization α and β appear only in the $(\alpha+\beta)/2$ and $(\alpha-\beta)/2$ combinations.

Let us consider first the PI case [10] defined by the differential equation $(z')^2(1-z^2)^{-1} = C$ (see table 1), which sets the third term on the right-hand side of (224) to a constant. Rewriting the first two terms as the function of z and rearranging the equation we get

$$\begin{aligned} E - V(x) &= C\left(n + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{C}{1-z^2(x)}\left[\frac{1}{4} - \left(\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2\right] \\ &\quad - \frac{2Cz(x)}{1-z^2(x)}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) . \end{aligned} \tag{225}$$

The $z(x)$ functions are the solutions of the differential equation defining the PI case, and their general form is given by the actual version of (10):

$$\int \frac{\mathrm{d}z}{(1-z^2)^{1/2}} = C^{1/2}x + \delta . \tag{226}$$

Depending on the nature of C (whether it is positive or negative) and that of z^2 (whether it is larger or smaller than 1), there are several solutions possible. In

[10] five different cases were identified, labeled by $z(x) = \mathrm{i}\sinh(ax)$, $\cosh(ax)$, $\cos(ax)$, $\cos(2ax)$ and $\cosh(2ax)$ for $C = -a^2$, $-a^2$, a^2, $4a^2$ and $-4a^2$. These correspond to the five PI type potentials listed in table 1. Also $z(x) = \sin(ax)$ is a solution, but it gives the same potential as $z(x) = \cos(ax)$, only shifted with π/a, therefore it was not considered as a separate solution in [10].

Let us now examine how these $z(x)$ functions behave under a $\mathcal{PT}$ transformation if we allow $\delta \neq 0$ in (226). The transformation properties of $z(x)$ also determine those of E and $V(x)$ in (225). It is easy to show that $\mathcal{PT}$ invariance of the potential cannot be reached in general if δ has a non-zero real component, because then the finite shift along the coordinate x renders $V(x)$ and its $\mathcal{PT}$ transformed version to essentially different forms. (There is an exception for those cases when $z(x)$ is a trigonometric function, because then the potentials are periodic. However, if we consider these potentials only within a single period, then $\mathcal{PT}$ invariance is lost for these special cases too.) If we set $\delta = \mathrm{i}\epsilon$, then the transformation properties of the $z(x)$ functions specified previously are the following:

$$\begin{aligned}
\mathcal{PT} : z(x) &= \mathrm{i}\sinh(ax + \mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = \mathrm{i}\sinh(ax + \mathrm{i}\epsilon) = z(x) \ , \\
\mathcal{PT} : z(x) &= \cosh(ax + \mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = \cosh(ax + \mathrm{i}\epsilon) = z(x) \ , \\
\mathcal{PT} : z(x) &= \cos(ax + \mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = \cos(ax + \mathrm{i}\epsilon) = z(x) \ , \\
\mathcal{PT} : z(x) &= \sin(ax + \mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = -\sin(ax + \mathrm{i}\epsilon) = -z(x) \ . \qquad (227)
\end{aligned}$$

The first three cases have been considered previously [10], while the fourth one has to be considered as a new independent possibility if we generalize our study to $\mathcal{PT}$ symmetric quantum mechanics.

The $\mathcal{PT}$ transformed version of (225) is

$$\begin{aligned}
\tilde{E} - \tilde{V}(x) &\equiv E^* - (V(-x))^* = \\
&= C\left(n + \frac{\alpha^* + \beta^* + 1}{2}\right)^2 - \frac{2C\tilde{z}(x)}{1 - \tilde{z}^2(x)}\left(\frac{\alpha^* + \beta^*}{2}\right)\left(\frac{\alpha^* - \beta^*}{2}\right) \\
&\quad + \frac{C}{1 - \tilde{z}^2(x)}\left[\frac{1}{4} - \left(\frac{\alpha^* + \beta^*}{2}\right)^2 - \left(\frac{\alpha^* - \beta^*}{2}\right)^2\right] \ . \qquad (228)
\end{aligned}$$

(Remember that we chose C to be real.) It is clear from (225) and (228) that $\mathcal{PT}$ invariance of the potential is satisfied for the $\tilde{z}(x) = z(x)$ cases if $(\alpha^*)^2 = \alpha^2$ and $(\beta^*)^2 = \beta^2$ holds, i.e. for $\alpha^* = \pm\alpha$ and $\beta^* = \pm\beta$. This can happen if α and β are purely real or imaginary.

When $\tilde{z}(x) = -z(x)$ holds, then the change of the sign in the last term in (228) has to be compensated with the appropriate choice of α and β. Requiring

Table 5: Conditions for having real and complex spectrum for $\mathcal{PT}$ symmetric shape-invariant potentials.

Type	$V(x)$	Real-energy regular solutions	Complex-energy regular solutions
PI	$-\frac{2(\alpha^2+\beta^2)-1}{4\cosh^2(x+\mathrm{i}\epsilon)} - \mathrm{i}\frac{(\alpha^2-\beta^2)\sinh(x+\mathrm{i}\epsilon)}{2\cosh^2(x+\mathrm{i}\epsilon)}$	α, β real $\epsilon \neq \frac{\pi}{2} \pm k\pi$	α or β imaginary, $\epsilon \neq \frac{\pi}{2} \pm k\pi$
	$\frac{2(\alpha^2+\beta^2)-1}{4\sinh^2(x+\mathrm{i}\epsilon)} + \frac{(\alpha^2-\beta^2)\cosh(x+\mathrm{i}\epsilon)}{2\sinh^2(x+\mathrm{i}\epsilon)}$	α, β real $\epsilon \neq k\pi$	α or β imaginary, $\epsilon \neq k\pi$
	$-\frac{4\beta^2-1}{4\cosh^2(x+\frac{\mathrm{i}}{2}\epsilon)} + \frac{4\alpha^2-1}{4\sinh^2(x+\frac{\mathrm{i}}{2}\epsilon)}$	α, β real $\epsilon \neq k\pi$	α or β imaginary, $\epsilon \neq k\pi$
	$\frac{2(\alpha^2+\beta^2)-1}{4\sin^2(x+\mathrm{i}\epsilon)} + \frac{(\alpha^2-\beta^2)\cos(x+\mathrm{i}\epsilon)}{2\sin^2(x+\mathrm{i}\epsilon)}$	α, β real $\epsilon \neq 0$	α and/or β imaginary $\mathrm{Im}(\alpha+\beta) \neq 0$, $\epsilon \neq 0$
	$\frac{4\beta^2-1}{4\cos^2(x+\frac{1}{2}\epsilon)} + \frac{4\alpha^2-1}{4\sin^2(x+\frac{1}{2}\epsilon)}$	α, β real $\epsilon \neq 0$	α and/or β imaginary $\mathrm{Im}(\alpha+\beta) \neq 0$, $\epsilon \neq 0$
	$\frac{2(\alpha^2+\beta^2)-1}{4\cos^2(x+\mathrm{i}\epsilon)} + \frac{(\alpha^2-\beta^2)\sin(x+\mathrm{i}\epsilon)}{2\cos^2(x+\mathrm{i}\epsilon)}$	$\beta = \alpha^*$ $\epsilon \neq 0$	$\beta = -\alpha^*$ imag., $\epsilon \neq 0$
PII	$-\frac{s(s+1)}{\cosh^2(x+\mathrm{i}\epsilon)} - 2\mathrm{i}\lambda\tanh(x+\mathrm{i}\epsilon)$	s, λ real $\epsilon \neq \frac{\pi}{2} \pm k\pi$	no such solutions
	$\frac{s(s+1)}{\sinh^2(x+\mathrm{i}\epsilon)} - 2\mathrm{i}\lambda\coth(x+\mathrm{i}\epsilon)$	s, λ real $\epsilon \neq k\pi$	no such solutions
	$\frac{s(s+1)}{\sin^2(x+\mathrm{i}\epsilon)} - 2\mathrm{i}\lambda\cot(x+\mathrm{i}\epsilon)$	s, λ real $\epsilon \neq 0$	$s = -\frac{1}{2} + \mathrm{i}\sigma$, $\epsilon \neq 0$
	$\frac{s(s+1)}{\cos^2(x+\mathrm{i}\epsilon)} + 2\mathrm{i}\lambda\tan(x+\mathrm{i}\epsilon)$	s, λ real $\epsilon \neq 0$	$s = -\frac{1}{2} + \mathrm{i}\sigma$, $\epsilon \neq 0$
LI	$\frac{\omega^2}{4}(x+\mathrm{i}\epsilon)^2 + (\alpha^2 - \frac{1}{4})\frac{1}{(x+\mathrm{i}\epsilon)^2}$	α real $\epsilon \neq 0$	α imaginary, $\epsilon \neq 0$

also $\mathcal{PT}$ invariance of the other potential term (which is an even function of $z(x)$) restricts the parameters to $(\alpha^*)^2 = \beta^2$, i.e. we get $\alpha^* = \pm\beta$. The $\alpha^* = \beta$ choice leads to $(\alpha + \beta)^* = \alpha + \beta$, and in this case the energy eigenvalues remain unchanged and are purely real. The $\alpha^* = -\beta$ choice also secures $\mathcal{PT}$ invariance of the potential, however, in this case the energy eigenvalues might become complex, due to $(\alpha + \beta)^* = -(\alpha + \beta)$.

We listed the individual PI type potentials and the corresponding energy formulae in table 5, along with the conditions for $\mathcal{PT}$ invariance. For the sake of completeness we also displayed the $z(x) = \cosh(2ax + \mathrm{i}\epsilon)$ and $z(x) = \cos(2ax + \mathrm{i}\epsilon)$ options, which are not independent cases, rather they can be obtained from the $z(x) = \cosh(ax + \mathrm{i}\epsilon)$ and $z(x) = \cos(ax + \mathrm{i}\epsilon)$ cases by the $a \to 2a$ replacement, using also formulae connecting hyperbolic and trigonometric functions with similar functions having half the original arguments. We also included in table 5 the $z(x) = \sin(ax + \mathrm{i}\epsilon)$ case, which did not appear in [10] as an independent problem, because $z(x) = \cos(ax)$ could be trivially obtained from $z(x) = \sin(ax)$ by a simple coordinate shift. As noted previously, real shifts of the coordinate are not compatible with $\mathcal{PT}$ invariance in general (e.g. $\delta = \mathrm{i}\epsilon$ is purely imaginary), therefore these two cases cannot be obtained from each other now, only if we define the potentials to be periodic.

The PI type potentials listed in table 5 are complex in general, due to the $\mathrm{i}\epsilon$ constant. If we set $\epsilon = 0$, the symmetric potential terms become real, while the odd ones turn purely imaginary. In fact, in this case the $z(x) = \cosh(ax + \mathrm{i}0)$ and the $z(x) = \cos(ax + \mathrm{i}0)$ potentials become fully real for any allowed α and β. The remaining two cases, $z(x) = \mathrm{i}\sinh(ax + \mathrm{i}0)$ and $z(x) = \sin(ax + \mathrm{i}0)$ present imaginary antisymmetric potential terms too.

There are also further special values of ϵ which deserve attention. It can be shown that the $\epsilon = 0$ version of the two hyperbolic PI type potentials can be obtained from the general $z(x) = \sinh(ax + \mathrm{i}\epsilon)$ and $z(x) = \cosh(ax + \mathrm{i}\epsilon)$ cases alike, when ϵ is set to $k\pi$ or $\pi/2 + k\pi$. This means that the two potentials can be continuously transformed into each other by carefully tuning ϵ.

According to (8) the solutions of the Schrödinger equation expressed in terms of a Jacobi polynomial have the form

$$\psi(x) \sim (1 - z(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + z(x))^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(z(x)) \ . \tag{229}$$

The regularity of these wavefunctions can be controlled by appropriate relations for α, β and n, whenever $|z| \to \infty$ or $z = \pm 1$ can occur. For the case of $|z| \to \infty$ regular behaviour of $\psi(x)$ can be guaranteed by the prescription $n + [\mathrm{Re}(\alpha + \beta) + 1]/2 < 0$. This condition sets an upper limit for n: $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$. When $z = 1$ or $z = -1$ can occur, then the regularity

of $\psi(x)$ requires $\mathrm{Re}(\alpha) > -1/2$ and $\mathrm{Re}(\beta) > -1/2$, respectively. Now let us see which of these conditions apply to the individual PI type potentials listed in table 5.

In the $z(x) = \mathrm{i}\sinh(ax + \mathrm{i}\epsilon)$ case only $|z| \to \infty$ has to be taken care of, for $x \to \pm\infty$. (We note that $z = \pm 1$ can also occur here if $\sin(\epsilon) = \pm 1$ holds, because in this case $z(x) \to \mp\cosh(ax)$. Since this special case corresponds to a particular example for the next PI type potential, we do not consider it here.) Then $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$ sets an upper limit for the number of bound states. This condition also means that there are no bound states for this potential if both α and β are imaginary. Special cases of this potential with $\epsilon = 0$ are mentioned in [83] ($\alpha = \frac{\lambda}{\mu} + 1$, $\beta = \frac{\lambda}{\mu} - 1$, $a = \mu$) and [84] ($\alpha = -b - A - \frac{1}{2}$, $\beta = b - A - \frac{1}{2}$, $a = 1$).

For $z(x) = \cosh(ax + \mathrm{i}\epsilon)$ the $n < -[\mathrm{Re}(\alpha + \beta) + 1]/2$ applies again, because $|z| \to \infty$ can occur. Now $z = \pm 1$ can also appear, if $\cos(\epsilon) = \pm 1$. In these cases a singularity appears in the potential at $x = 0$. (We note that this singularity also appears for the conventional version of this potential, which is considered as a radial problem.) If we exclude these particular values of ϵ, then the potential becomes finite everywhere, and there are no further restrictions for the potential parameters. Similarly to the previous case, there are no bound states if both α and β are imaginary. The $\mathcal{PT}$ symmetric Pöschl–Teller potential discussed in [87] corresponds to this case, taking $\alpha = -A - 1/2$, $\beta = B - 1/2$, $C = -4$, $a = 1$ and using -2ϵ instead of ϵ.

In the trigonometric cases $z(x) = \cos(ax + \mathrm{i}\epsilon)$ and $z(x) = \sin(ax + \mathrm{i}\epsilon)$, $|z| \to \infty$ cannot occur, therefore no conditions limit the possible values of n. Furthermore, $z = \pm 1$ can also occur for $\epsilon = 0$ only, in which case these potentials have singularities at $ax = k\pi$, and $ax = (k + \frac{1}{2}\pi)$, respectively, similarly to the conventional versions of these problems. Then the $\mathrm{Re}(\alpha) > -1/2$ and $\mathrm{Re}(\beta) > -1/2$ conditions also have to be observed in both cases, and have to be combined with the other conditions for α and β required by $\mathcal{PT}$ symmetry.

The regularization of the potentials by eliminating their singularities with appropriate choices of ϵ relaxes the boundary conditions considerably. This means that in principle, the second independent solution of the Schrödinger equation (which is disqualified due to these boundary conditions for the conventional problems [143]) also becomes allowed. In fact, the general solution of the Schrödinger equation can then be written in terms of two hypergeometric functions as

$$\psi(x) \sim \qquad (1-y)^{\frac{1}{2}(a+b-c)+\frac{1}{4}} \left(C_1 y^{\frac{2c-1}{4}} F(a,b;c;y) \right.$$

$$+C_2 y^{\frac{3-2c}{4}} F(b-c+1, a-c+1; 2-c; y)\Big) \ , \qquad (230)$$

where $y = (1-z(x))/2$. The particular solution (229) can be obtained from (230) by setting $C_2 = 0$, $a = -n$, $b = n+\alpha+\beta+1$ and $c = \alpha+1$, which reduces the remaining hypergeometric function to a Jacobi polynomial. (See equation 22.5.42 in [19].) As we shall see in subsection 3.4.2, the second solution does not introduce anything essentially new, rather it corresponds to changing α to $-\alpha$.

Let us turn to the PII case [10] defined by the differential equation $(z')^2(1-z^2)^{-2} = C$, in which case the fourth term on the right-hand side of (89) becomes a constant:

$$\begin{aligned} E - V(x) &= -2C\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) z(x) - C\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2\right] \\ &\quad + C\left(n + \frac{\alpha+\beta}{2}\right)\left(n + \frac{\alpha+\beta}{2} + 1\right)(1 - z^2(x)) \ . \end{aligned} \qquad (231)$$

With a parameter transformation the n-dependence can be transferred to the constant (energy) term. The potential then can be written as

$$V(x) = -Cs(s+1)(1-z^2(x)) - 2C\Lambda z(x) \ , \qquad (232)$$

where $s = n + \frac{(\alpha+\beta)}{2}$ or $s = -n - \frac{(\alpha+\beta)}{2} - 1$ and $\Lambda = \frac{\alpha-\beta}{2}\frac{\alpha+\beta}{2}$. This gives $\alpha = s-n+\Lambda/(s-n)$, $\beta = s-n-\Lambda/(s-n)$, or $\alpha = -s-n-1-\Lambda/(s+n+1)$, $\beta = -s-n-1+\Lambda/(s+n+1)$. The energy eigenvalues are then given by $E = -C\left((s-n)^2 + \frac{\Lambda^2}{(s-n)^2}\right)$ or $E = -C\left((s+n+1)^2 + \frac{\Lambda^2}{(s+n+1)^2}\right)$. In order to simplify the formalism, in what follows we consider only the first set of the above relations: the second set can be obtained by the $s \to -s-1$ substitution.

The $z(x)$ functions are again supplied by the current version of (10):

$$\int \frac{\mathrm{d}z}{1-z^2} = C^{1/2}x + \delta \ . \qquad (233)$$

In [10] the $\delta = 0$ choice was made and three independent solutions were identified: $z = \tanh(ax)$, $\coth(ax)$ and $-\mathrm{i}\cot(ax)$ with $C = a^2$, a^2 and $-a^2$, respectively. One further solution, $z = \mathrm{i}\tan(ax)$ with $C = -a^2$ is essentially the same as the $-\mathrm{i}\cot(ax)$ case, therefore it was not discussed as a separate possibility.

Considering the $\mathcal{PT}$ symmetric case, we again find that the $\delta = \mathrm{i}\epsilon$ choice has to be made in order to reach $\mathcal{PT}$ invariance of the potentials. The transformation properties of the four possible $z(x)$ functions under the $\mathcal{PT}$ operation

are the following:

$$\begin{aligned}
\mathcal{PT}: z(x) &= \tanh(ax+\mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = -\tanh(ax+\mathrm{i}\epsilon) = -z(x)\ , \\
\mathcal{PT}: z(x) &= \coth(ax+\mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = -\coth(ax+\mathrm{i}\epsilon) = -z(x)\ , \\
\mathcal{PT}: z(x) &= -\mathrm{i}\cot(ax+\mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = -\mathrm{i}\cot(ax+\mathrm{i}\epsilon) = z(x)\ , \\
\mathcal{PT}: z(x) &= \mathrm{i}\tan(ax+\mathrm{i}\epsilon) \longrightarrow \tilde{z}(x) = \mathrm{i}\tan(ax+\mathrm{i}\epsilon) = z(x)\ . \qquad (234)
\end{aligned}$$

Similarly to the PI case, the last $z(x)$ function can be obtained from the other trigonometric one by using $\delta = -\pi/(2a) + \mathrm{i}\epsilon$ instead of $\delta = \mathrm{i}\epsilon$. However, we again considered it an independent case because finite real translations are not compatible with $\mathcal{PT}$ invariance in general. The $\mathcal{PT}$ transformed potential (232) becomes

$$\tilde{V}(x) = -Cs^*(s^*+1)(1-\tilde{z}^2(x)) - 2C\Lambda^*\tilde{z}(x) \qquad (235)$$

and the corresponding energy eigenvalues are $\tilde{E} = -C\left((s^*-n)^2 + \frac{(\Lambda^*)^2}{(s^*-n)^2}\right)$. For the $\tilde{z}(x) = z(x)$ cases $\mathcal{PT}$ invariance is reached if $(s(s+1))^* = s(s+1)$ and $\Lambda^* = \Lambda$. This means that Λ has to be real, while s is either real, or $s = -\frac{1}{2} + \mathrm{i}\sigma$. In the first case α and β are both real, and the energy eigenvalues are also real. When $\tilde{z}(x) = -z(x)$, then for $\mathcal{PT}$ invariance we need $(s(s+1))^* = s(s+1)$ and $\Lambda^* = -\Lambda$. In this case Λ has to be imaginary, while s can have the same values as in the previous case. If s is real, then $\alpha^* = \beta$ holds, and the energy eigenvalues are real. The general form of the solutions is now

$$\psi(x) \sim (1-z(x))^{\frac{\alpha}{2}}(1+z(x))^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(z(x))\ . \qquad (236)$$

Table 5 contains the individual PII type potentials, the energy formulae, and the conditions for $\mathcal{PT}$ invariance. Their detailed analysis can be performed similarly to the procedure presented in the PI case: the $\mathcal{PT}$ symmetry of the potential, the normalizability of the wavefunctions and the reality of the energy spectrum determine conditions for the potential parameters.

For the generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ [19] the current form of (6) becomes (71) [P9], and the corresponding solutions, according to (8) are

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}}(z(x))^{\frac{\alpha+1}{2}} \exp(-z(x)/2) L_n^{(\alpha)}(z(x))\ . \qquad (237)$$

Picking the third term on the right-hand side of (71) as a constant and setting $(z')^2 z^{-1} = C$ we get the LI case [10]. Equation (71) can be rewritten as

$$E - V(x) = C\left(n + \frac{\alpha+1}{2}\right) - \frac{C}{4}z(x) - \frac{C}{4z(x)}\left(\alpha+\frac{1}{2}\right)\left(\alpha-\frac{1}{2}\right)\ . \qquad (238)$$

According to (10) the solution of the defining differential equation of $z(x)$ is given by

$$z(x) = \frac{C}{4}(x + \bar{\delta})^2 \tag{239}$$

with $\bar{\delta} = \delta/C^{1/2}$. Again we find that only the $\bar{\delta} = \mathrm{i}\epsilon$ choice with real ϵ can result in a $\mathcal{PT}$ invariant potential and that the $\mathcal{PT}$ transform of $z(x)$ is

$$\mathcal{PT} : z(x) = \frac{C}{4}(x + \mathrm{i}\epsilon)^2 \longrightarrow \tilde{z}(x) = \left(\frac{C}{4}(-x + \mathrm{i}\epsilon)^2\right)^* = \frac{C}{4}(x + \mathrm{i}\epsilon)^2 = z(x) \ . \tag{240}$$

The $\mathcal{PT}$ transform of (238) is

$$E^* - (V(-x))^* = C\left(n + \frac{\alpha^* + 1}{2}\right) - \frac{C}{4}\tilde{z}(x) - \frac{C}{4\tilde{z}(x)}\left(\alpha^* + \frac{1}{2}\right)\left(\alpha^* - \frac{1}{2}\right) \ . \tag{241}$$

Comparing (238) and (241) we find that $\mathcal{PT}$ symmetry holds if $(\alpha^2)^* = \alpha^2$ is satisfied, i.e. if α is purely real or imaginary ($\alpha^* = \pm\alpha$). In the former case the energy eigenvalues will be real, despite the complex potential terms.

By using (237) a particular solution of the corresponding Schrödinger equation is written as

$$\psi(x) \sim (z(x))^{\frac{2\alpha+1}{4}} \exp(-z(x)/2) L_n^{(\alpha)}(z(x)) \ . \tag{242}$$

Since $\mathrm{Re}(z) > 0$ for $x \to \infty$ (and also for $x \to -\infty$), the solutions vanish asymptotically.

In the conventional treatment of this problem [10] the $\bar{\delta} = 0$ choice was made in order to obtain a radial problem defined on the positive semi-axis. The $C = 2\omega > 0$ choice rendered the energy to be positive, and $\alpha = l + 1/2$ accounted for the centrifugal term. This means that the wavefunction behaves like x^{l+1} near the origin. Solutions which are non-zero at the origin are not considered physical in the conventional case, when solutions only on the positive semi-axis are taken into account. However, in the $\mathcal{PT}$ symmetric case the singularity represented by the centrifugal barrier vanishes if $\epsilon \neq 0$ holds, therefore the problem can (and should) be extended to the full x axis. In this case the general solution of the problem can be written in terms of two confluent hypergeometric functions [19], but similarly to the PI and PII cases it turns out that the second set of solutions are obtained from the first one by the $\alpha \to -\alpha$ replacement. This has been discussed in [86], where the $\mathcal{PT}$ symmetric harmonic oscillator was introduced, and this possible double sign of α has been attributed to a "quasi-parity" quantum number. The parameters used there are related to the present ones via $C = 4$, $\alpha^2 - 1/4 = G$ and $\epsilon = -c$.

We note that this extension of the radial problem to the full line in the $\mathcal{PT}$ symmetry context also contains the one-dimensional harmonic oscillator. In that case the centrifugal barrier does not appear, which corresponds to setting α to $1/2$ and $-1/2$. The generalized Laguerre polynomials then reduce to Hermite polynomials, which are odd and even, respectively, corresponding to the odd and even solutions of the one-dimensional problem. In the $\mathcal{PT}$ symmetric context there is no point in discussing the Hermite polynomials and the one-dimensional harmonic oscillator separately, as in [10] for ordinary quantum mechanics.

When we attempt to analyze the LII (Coulomb) and LIII (Morse) cases in the $\mathcal{PT}$ symmetric context by solving the differential equations $(z')^2 = C$ and $(z')^2 z^{-2} = C$ (as in [10]), we arrive at limits of applicability of the present approach. Their nature is clearly visible from the form of the general solutions (237): the normalizability of the wave functions does not depend on the powers of the various terms in (237) as for Jacobi polynomials, rather the boundary conditions are determined by the $z(x)$ function itself, which appears in an exponent. In particular, one should have $z \to \infty$ for $x \to \pm\infty$ to secure normalizability of the wavefunctions. This was guaranteed in the LI (harmonic oscillator case) by the form of $z(x)$ in (239), but the corresponding solutions in the Coulomb and the Morse cases, i.e. $z(x) = C^{1/2}x + \delta$ and $z(x) = \exp(C^{1/2}x + \delta)$ lead to infinities at one limit. In conventional quantum mechanics the latter two cases are considered as radial problems, therefore it is enough to have regularity of $z(x)$ for $x \to \infty$. A way around this problem can be found if one replaces the linear integration path $(x + \mathrm{i}\epsilon)$ with curved ones. One possible way to find such curved integration paths is to apply a variable transformation to the $\mathcal{PT}$ symmetric harmonic oscillator problem [86] to get $\mathcal{PT}$ symmetric Morse [91] and Coulomb [P8] potentials.

A similar analysis was also made to derive conditions under which the same potentials have complex energy eigenvalues [P13]. The results are also displayed in table 5. In some cases the conditions regarding $\mathcal{PT}$ symmetry of the potential, normalizability of the eigenfunctions and the complex nature of the energy eigenvalues contradicted each other, so it turned out that certain potentials cannot have normalizable states at complex energy eigenvalues. The results in [P13] agree with those of [144] for the harmonic oscillator, and contain the findings of [93] on the Scarf II potential as a special case, but besides these they were all new.

A general feature of the shape-invariant potentials discussed here is that the functional form of the potentials depends on the *squares* of the potential parameters which can take on imaginary values (i.e. α, β, $\mathrm{i}\sigma$), therefore the

potentials are insensitive to the sign of this parameter. However, this sign appears explicitly in the energy formulae as the sign of the imaginary component of the energy, thus the occurrence of complex conjugate energy pairs is a necessity. From the structure of the energy formulae it is apparent that depending on the potential parameters, the energy eigenvalues of these potentials are either all real or complex, so they practically do not occur together at the same time.

> It is worth noting that besides the shape-invariant case, the present method also works for some "implicit" potentials, such as the Ginocchio potential [13]. As discussed in subsection 3.1.2, this is obtained by setting $\alpha = \beta$ in (224), reducing the Jacobi polynomial to an ultraspherical (or Gegenbauer) one [19], and considering (100) in (9) [C3]. It can be shown without deeper analysis that the Ginocchio potential can be made $\mathcal{PT}$ symmetric by the present method. In particular, the actual form of (10) becomes [C3]
>
> $$\delta^{\frac{1}{2}} \tanh^{-1}\left(z\delta^{\frac{1}{2}}(\delta+1-z^2)^{-\frac{1}{2}}\right)+\tan^{-1}\left(z(\delta+1-z^2)^{-\frac{1}{2}}\right) = C^{\frac{1}{2}}x+\mathrm{i}\epsilon\ , \quad (243)$$
>
> and even this implicit functional form shows (e.g. via a series expansion) that the $\mathcal{PT}$ transform of $z(x)$, $\tilde{z}(x) \equiv (z(-x))^* = -z(x)$, therefore $V(x)$, in which $z(x)$ appears only through $z^2(x)$ (see also (102) and [C3]) must be $\mathcal{PT}$ invariant. For potentials beyond the Natanzon class, one has to check each case individually. There the $F(z)$ function can have more general forms, and it is not guaranteed that it satisfies a second-order differential equation as in our approach.

Finally, we note that the imaginary coordinate shift can be interpreted as a Hermitian linear automorphism defining η-pseudo-Hermiticity. In this case $\eta = \exp(\epsilon p)$, where p is the momentum operator $p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$, and $\eta^{-1}V(x)\eta = V(x+\mathrm{i}\epsilon)$ readily follows [145].

3.4.2 An illustration: the $\mathcal{PT}$ symmetric Scarf II potential

Here we consider the $\mathcal{PT}$ symmetric Scarf II potential to illustrate the unusual features of $\mathcal{PT}$ symmetry [P16], including the case of unbroken and spontaneously broken $\mathcal{PT}$ symmetry, the appearance of the quasi-parity quantum number q associated with the richer bound-state energy spectrum, the modified inner product and the pseudo-norm derived from it, which is known to have indefinite sign.

The $\mathcal{PT}$ symmetric Scarf II potential occupies a special position among $\mathcal{PT}$ symmetric shape-invariant potentials. It is defined on the whole x axis, it

has no singularity at $x = 0$, and in contrast with most other shape-invariant potentials, it can be turned into a $\mathcal{PT}$ symmetric form without regularizing its singularity by means of an $x \to x + \mathrm{i}\epsilon$ imaginary coordinate shift [P9, 145]. Therefore it is not surprizing that it became a "guinea pig" of testing $\mathcal{PT}$ symmetry on a solvable example. It has been associated with the sl(2,C) [146], su(1,1)$\simeq$so(2,1) [P10] and so(2,2) [P15] potential algebras, and it has also been observed that its $\mathcal{PT}$ symmetric version has a second set of bound states, which appear as resonances in its Hermitian version [146, P10]. This mechanism of doubling the bound states is essentially different from the one arising from the cancellation of singularities at $x = 0$ by the imaginary coordinate shift. This potential is also known to have (purely) real and (purely) complex energy spectrum, depending on the relative strength of its real and imaginary component [93], and since the two domains can be connected with a continuous tuning of the parameters without crossing a singularity, it is a perfect example to illustrate the breakdown of $\mathcal{PT}$ symmetry.

Here we follow the notation applied throughout the present subsection and also used in [P9, P13, P16] to discuss the Scarf II potential

$$V(x) = -\frac{1}{\cosh^2(x)}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2\mathrm{i}\sinh(x)}{\cosh^2(x)}\left(\frac{\beta+\alpha}{2}\right)\left(\frac{\beta-\alpha}{2}\right) . \tag{244}$$

The bound-state energy eigenvalues are

$$E_n^{(\alpha,\beta)} = -\left(n + \frac{\alpha+\beta+1}{2}\right)^2 , \tag{245}$$

while the corresponding wavefunctions

$$\psi_n^{(\alpha,\beta)}(x) = C_n^{(\alpha,\beta)}(1 - \mathrm{i}\sinh(x))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh(x))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x)) \tag{246}$$

are expressed in terms of Jacobi polynomials [19] and are normalizable if $n < -[\mathrm{Re}(\alpha+\beta)+1]/2$ holds.

In the Hermitian case α and β are complex and satisfy $\alpha^* = \beta$: $\alpha = -s - \frac{1}{2} - \mathrm{i}\lambda$, $\beta = -s - \frac{1}{2} + \mathrm{i}\lambda$ [147, 10]. In this case only one regular solution exists. It is obvious that with arbitrary α and β the general complex version of the Scarf II potential is obtained.

As discussed previously [P9, P13], the Scarf II potential can be made $\mathcal{PT}$ symmetric if $\alpha^* = \pm\alpha$ and $\beta^* = \pm\beta$ holds [P9], i.e. if α and β are both either real or imaginary. In order to have real energy eigenvalues both α and β have to be real, while to have complex bound state spectrum, i.e. in the case of

spontaneous breakdown of $\mathcal{PT}$ symmetry *one of them* has to take an imaginary value [P13]. If both α and β are imaginary, then there are no bound states. Here we assume that β is real, and α can be real or imaginary, depending on whether the $\mathcal{PT}$ symmetry is unbroken or broken. This choice does not restrict the generality of the problem, since the roles of α and β can easily be reversed, due to the properties of the Jacobi polynomials [19].

For the Scarf II potential the breakdown of $\mathcal{PT}$ symmetry takes place when the strength of the imaginary potential component exceeds a certain limit depending on the strength of the real potential component, as described in [93]. This condition corresponds exactly to taking imaginary values for α instead of real ones (see e.g. [P13] for the details), so a smooth transition over the critical point can be achieved by moving α to zero along the real axis and then continuing along the imaginary axis.

In the $\mathcal{PT}$ symmetric case there are two sets of normalizable solutions [P9, 146, P10], which carry the upper indexes (α, β) and $(-\alpha, \beta)$ in (246). Obviously, (244) is not sensitive to the $+$ or $-$ sign of α. In the notation of [148] the two solutions corresponds to quasi-parity $q = +1$ and -1. In what follows, therefore, α can implicitly be replaced with $q\alpha$. This sign difference results in two distinct energy eigenvalues in (245), which form a complex conjugate pair when α is imaginary, i.e. in the case of broken $\mathcal{PT}$ symmetry. In this case the $\mathcal{PT}$ operation transforms the two solutions into each other, while in the unbroken symmetry case the two solutions are eigenfunctions of the $\mathcal{PT}$ operator.

In what follows we are going to evaluate integrals containing the standard and $\mathcal{PT}$ symmetric inner product of wavefunctions of the type $\psi_n^{(\pm\alpha,\beta)}(x)$ (246). Let us consider the $\mathcal{PT}$ symmetric inner product [95, 96] of two solutions of the type (246)

$$I_{nl}^{(\alpha,\beta,\delta)} = \int_{-\infty}^{\infty} \psi_n^{(\alpha,\beta)}(x)[\psi_l^{(\delta,\beta)}(-x)]^* \mathrm{d}x \ . \tag{247}$$

According to our choice, β is real and δ can be $\pm\alpha$, depending on whether we calculate the PT symmetric inner product of states with the same or different quasi-parities ($\delta = \alpha$ and $\delta = -\alpha$, respectively), furthermore, α can be real or imaginary, depending on whether the PT symmetry is unbroken or broken. Without presenting the technical details [P16], we just state the result

$$I_{nl}^{(\alpha,\beta,\delta)} = C_n^{(\alpha,\beta)}[C_l^{(\delta,\beta)}]^*(-1)^n Q_{nl}^{(\alpha,\beta,\beta,\delta)} \ , \tag{248}$$

which can be evaluated as the special case of

$$Q_{nl}^{(\alpha,\beta,\gamma,\delta)} = (-1)^{n+l} 2^{\frac{\alpha+\beta+\gamma^*+\delta^*}{2}+2} \frac{\sin[\pi(\alpha+\delta^*)/2]\sin[\pi(\beta+\gamma^*)]}{\sin[\pi(\alpha+\beta+\gamma^*+\delta^*)/2)]}$$

$$\times \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n+\alpha \\ m \end{array} \right) \left(\begin{array}{c} n+\beta \\ n-m \end{array} \right) \sum_{m'=0}^{l} (-1)^{m'} \left(\begin{array}{c} l+\gamma^* \\ m' \end{array} \right) \left(\begin{array}{c} l+\delta^* \\ l-m' \end{array} \right)$$
$$\times \frac{\Gamma(\frac{\alpha+\delta^*}{2}+n-m+m'+1)\Gamma(\frac{\beta+\gamma^*}{2}+l+m-m'+1)}{\Gamma(\frac{\alpha+\beta+\gamma^*+\delta^*}{2}+n+l+2)} \ . \qquad (249)$$

with $\gamma = \beta$. This closed formula can be obtained in a multistep way by substituting in (246) the explicit expression

$$P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x)) = \frac{1}{2^n} \sum_{m=0}^{n} \left(\begin{array}{c} n+\alpha \\ m \end{array} \right) \left(\begin{array}{c} n+\beta \\ n-m \end{array} \right)$$
$$\times (-1)^{n-m} (1-\mathrm{i}\sinh(x))^{n-m} (1+\mathrm{i}\sinh(x))^m \ , \qquad (250)$$

then evaluating the integrals of the type

$$A_i^{(s,t)} \equiv \int_{-\infty}^{\infty} (1-\mathrm{i}\sinh x)^s (1+\mathrm{i}\sinh x)^t (\sinh x)^i \mathrm{d}x \qquad i = 0, 1 \ , \qquad (251)$$

and finally calculating the double sum for the running indices. As discussed in [P16], the complicated expression of binomial coefficients reduces significantly whenever $\delta^* = \alpha$ and $\gamma^* = \beta$ holds, and this actually happens not only for the case of the $\mathcal{PT}$ symmetric inner product, but also for the usual one, in case of the Hermitian Scarf II potential.

Equation (248) together with (249) has significant implications regarding the $\mathcal{PT}$ symmetric inner product (247). First note that whenever $\alpha = -\delta^*$ holds, the integral vanishes due to the presence of the $\sin[\pi(\alpha+\delta^*)]$ term in (249). This corresponds to either $\alpha = \delta$ with imaginary α, i.e. the inner product of wavefunctions of the same type (same quasi-parity) in the broken $\mathcal{PT}$ symmetry case, or $\alpha = -\delta$ with real α, i.e. the inner product of two different type (different quasi-parity) wavefunctions in the unbroken $\mathcal{PT}$ symmetry case. So we can conclude that the two states are orthogonal in these situations.

Let us consider the cases when $\alpha \neq -\delta^*$. The first case is $\delta = \alpha$ with real α (unbroken PT symmetry). With this choice (and remembering that β is real) we get

$$I_{nl}^{(\alpha,\beta,\alpha)} = \delta_{nl} |C_n^{(\alpha,\beta)}|^2 \frac{2^{\alpha+\beta+2}}{\alpha+\beta+2n+1} \frac{\sin(\pi\alpha)\sin(\pi\beta)}{\sin[\pi(\alpha+\beta)]}$$
$$\times \left(\begin{array}{c} \alpha+\beta+2n \\ n+\beta \end{array} \right)^{-1} \left(\begin{array}{c} \alpha+\beta+2n \\ n \end{array} \right) \ . \qquad (252)$$

This proves directly the orthogonality of the states of the same type (i.e. those with the same quasi-parity) for $n \neq l$ when the PT symmetry is unbroken, and gives a closed formula for the pseudonorm for $n = l$.

Previously this pseudonorm was known only for the ground state $n = 0$ [148], while the orthogonality of the eigenfunctions was proven only indirectly [96, 93]. This latter proof rests on the equation

$$(E_n - E_l^*) \int_{-\infty}^{\infty} \psi_n(x)\psi_l^*(-x)\mathrm{d}x = 0 \ , \tag{253}$$

which is the equivalent of the equation proving the real nature of the energy eigenvalues for Hermitan systems. In the case of unbroken $\mathcal{PT}$ symmetry E_n and E_l are real and they are not equal, consequently the integral in (253) has to vanish.

The only remaining case is $\delta = -\alpha$ with imaginary α, when $\delta^* = \alpha$ holds again. This case gives us the overlap of eigenstates belonging to different quasi-parity in the broken PT symmetry case. It turns out that the $I_{nl}^{(\alpha,\beta,-\alpha)}$ overlap has the same form as (252), except that $|C_n^{(\alpha,\beta)}|^2$ has to be replaced with $C_n^{(\alpha,\beta)}[C_l^{(-\alpha,\beta)}]^*$.

Let us summarize the results for the different cases [P16].

- Unbroken $\mathcal{PT}$ symmetry (α real), same quasi-parities: $I_{nl}^{(\alpha,\beta,\alpha)}$ is diagonal in n and l, as seen from (252). To extract more information, we can rewrite equation (252) in a somewhat different form, after eliminating the sine functions from the formulae by combining them with some gamma functions via $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$:

$$I_{nl}^{(\alpha,\beta,\alpha)} = \delta_{nl}(-1)^n\pi|C_n^{(\alpha,\beta)}|^2 \frac{2^{\alpha+\beta+2}}{(-\alpha-\beta-2n-1)n!} \frac{\Gamma(-\alpha-\beta-n)}{\Gamma(-\alpha-n)\Gamma(-\beta-n)} \ . \tag{254}$$

  Due to the condition for having bound states, i.e. $n < -[\mathrm{Re}(\alpha)+\beta+1]/2$, if α is real, every term in (254) is positive, except $(-1)^n$ which alternates, and $[\Gamma(-\alpha-n)\Gamma(-\beta-n)]^{-1}$, which is real, but its sign depends on the relative magnitude of α, β and n. Except for extreme values of α and β the argument of the two gamma functions is positive for the first few n's, so then the alternating $(-1)^n$ factor determines the sign of the pseudonorm, but as n reaches $-\alpha$ and/or $-\beta$, this regular pattern changes. The results concerning this case are new, except for $n = 0$.

- Unbroken $\mathcal{PT}$ symmetry (α real), different quasi-parities: $I_{nl}^{(\alpha,\beta,-\alpha)} = 0$, due to $\sin\pi(\alpha - \alpha^*) = 0$ in (249). This was already proven though indirectly by (253) [96, 93].

- Broken $\mathcal{PT}$ symmetry (α imaginary), same quasi-parities: $I_{nl}^{(\alpha,\beta,\alpha)} = 0$, due to $\sin \pi(\alpha - \alpha^*) = 0$ in (249). This was already proven though indirectly by (253) [96, 93].

- Broken $\mathcal{PT}$ symmetry (α imaginary), different quasi-parities: $I_{nl}^{(\alpha,\beta,-\alpha)}$ is diagonal in n and l, as seen from (252). But in this case it seems that for $n = l$ there can be two *different* wavefunctions which are **not** orthogonal, in general. Equation (254) holds for this case too, except for a change in the term containing the normalization constants, as discussed before. This non-orthogonality of two different states is a new feature of $\mathcal{PT}$ symmetric problems, which in this case appears only when the $\mathcal{PT}$ symmetry is broken. This unusual new result seems to be supported by (253): when the $\mathcal{PT}$ symmetry is broken, the energies of the two states with the same principal quantum number n but with different quasi-parity are complex conjugate to each other, so the zero value of (253) is secured by the energy term, and the integral need not be zero.

This completes the analysis of the possible integrals of the type (247).

For the sake of completeness we also present a side product of our calculations: the normalization coefficients calculated for the bound-state wavefunctions of the Hermitian Scarf II potential. These can be determined from

$$\int_{-\infty}^{\infty} \psi_n^{(\alpha,\beta)}(x)[\psi_l^{(\alpha,\beta)}(x)]^* \mathrm{d}x = C_n^{(\alpha,\beta)}[C_l^{(\alpha,\beta)}]^* Q_{nl}^{(\alpha,\beta,\alpha,\beta)} \ , \tag{255}$$

because in this case $\beta^* = \alpha$ also simplifies the summation in (249), and we find that

$$C_n^{(\alpha,\beta)} = 2^{-\frac{\alpha+\beta}{2}-1} \left[\frac{\Gamma(-\alpha-n)\Gamma(-\beta-n)(-\alpha-\beta-2n-1)n!}{\Gamma(-\alpha-\beta-n)\pi} \right]^{1/2} \ . \tag{256}$$

These normalization coefficients have not been known previously due to the involved mathematics [147, 93].

Finally, we note that an expression similar to (159) connecting the imaginary component of the potential with that of the energy can also be evaluated for the sake of tracing the mechanism of the spontaneous breaking of the $\mathcal{PT}$ symmetry as α is moved from real to imaginary values through $\alpha = 0$. Unfortunately, in this case the double sums in (249) remain rather complicated, nevertheless, the integrals can be evaluated for the first few states, and we indeed get

$$\mathrm{Im}(E_n^{(\alpha,\beta)}) = \frac{\mathrm{i}}{8}(\alpha - \alpha^*)(\alpha + \alpha^* + 2\beta + 4n + 2) \ , \tag{257}$$

as expected from (245).

We also mention a *mathematical "byproduct"* of the present calculations, i.e. a formula missing from the standard compilations [149]:

$$\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} a-m-k \\ n-m \end{array} \right) \left(\begin{array}{c} b+m+k \\ m \end{array} \right) = (-1)^m \left(\begin{array}{c} n \\ m \end{array} \right) , \qquad (258)$$

which can be proven by recursion. The interesting feature of this result is that the right-hand side is independent of a and b.

Besides the bound states, the scattering states have also been analyzed for the Scarf II potential, and the transmission and reflection coefficients have been determined [P10]. In this analysis the more general version of the Scarf II potential was considered, with the $x \to x + \mathrm{i}\epsilon$ imaginary coordinate shift. The transmission and reflection coefficients of the Scarf II potential were found to be

$$\begin{aligned} T(k,\alpha,\beta) &= \frac{\Gamma(\frac{1}{2}(\alpha+\beta+1)-\mathrm{i}k)\Gamma(-\frac{1}{2}(\alpha+\beta-1)-\mathrm{i}k)}{\Gamma(-\mathrm{i}k)\Gamma(1-\mathrm{i}k)\Gamma^2(\frac{1}{2}-\mathrm{i}k)} \\ &\quad \times\Gamma(\frac{1}{2}(\beta-\alpha+1)-\mathrm{i}k)\Gamma(\frac{1}{2}(\alpha-\beta+1)-\mathrm{i}k) , \qquad (259) \end{aligned}$$

$$\begin{aligned} R(k,\alpha\beta) &= \mathrm{i}\exp(2\epsilon k)\left(\frac{\cos(\frac{\pi}{2}(\alpha+\beta+1))\sin(\frac{\pi}{2}(\alpha-\beta))}{\cosh(\pi k)}\right. \\ &\quad \left. -\frac{\sin(\frac{\pi}{2}(\alpha+\beta+1))\cos(\frac{\pi}{2}(\alpha-\beta))}{\sinh(\pi k)}\right) T(k,\alpha,\beta) . \qquad (260) \end{aligned}$$

These equations contain both the Hermitian case with $\epsilon = 0$ and $\alpha = \beta^* = -s - \frac{1}{2} - \mathrm{i}\lambda$ [P10, 147], and the $\mathcal{PT}$ symmetric one with unbroken (α real) or spontaneously broken (α imaginary). It is remarkable that the imaginary shift affects only the reflection coefficient (260). This exponential factor clearly shows that the $|T|^2 + |R|^2 = 1$ relation breaks down in the $\mathcal{PT}$ symmetric case, which is not surprizing if we recall that we have complex potentials in this case, in which the flux is not conserved. We note that although the extra $\exp(2\epsilon k)$ factor increases the modulus of the reflection amplitude (260) if $\epsilon k > 0$, it remains finite as long as $\epsilon < \pi/2$. Since $\cosh(x + \mathrm{i}\frac{\pi}{2}) = \mathrm{i}\sinh(x)$, for this particular value of ϵ the potential becomes [P9] a singular (generalized Pöschl–Teller) potential, and equations (259) and (260) do not apply.

3.4.3 Other types of $\mathcal{PT}$ symmetric potentials

A large number of $\mathcal{PT}$ symmetric potentials have to be defined along bent contours of the complex x plane in order to generate normalizable solutions. This

was the case with the first examples of $\mathcal{PT}$ symmetric potentials [89] which were found numerically, but there are also exactly solvable potentials with this property. It is not surprizing that defining potentials on bent contours of the x plane introduces further exotic features of $\mathcal{PT}$ symmetric potentials, making them more interesting for mathematical, rather than physical investigations, nevertheless, here we mention some examples for the sake of completeness.

As we have seen in subsection 3.4.1, the imaginary coordinate shift failed to turn the Morse and the Coulomb potentials into $\mathcal{PT}$ symmetric problems, because the solutions were not normalizable along the $x+\mathrm{i}\epsilon$ line. This transformation, actually, takes the Morse potential into a non-$\mathcal{PT}$-symmetric problem, which, however, has real energy eigenvalues [146, 145]. Alternatively, its $\mathcal{PT}$ symmetric version has to be defined along a bent contour [91]. The situation is similar for the $\mathcal{PT}$ symetric Coulomb potential too, which can most conveniently introduced by applying a variable transformation to the $\mathcal{PT}$ symmetric harmonic oscillator [86]. The Liouvillean method [118] (or the point canonical transformation [117]) presented in subsection 3.1.3 offers a convenient framework for this. This operation can also be recognized as the $\mathcal{PT}$ symmetric Kustaanheimo–Steifel transformation [P8].

We start with establishing the notation for the $\mathcal{PT}$ symmetric harmonic oscillator, introducing also explicitly the quasi-parity quantum number. (See subsection 3.4.1 and [86].) The potential is defined as in table 5, except that for simplicity we put $\omega = 2$. Then the energy eigenvalues are $E_{(n,q)} = 4n + 2 - 2\,q\alpha$, now exhibiting the q quantum number too, while the corresponding normalizable eigenfunctions can be written (see also (242)) as

$$\psi_{(n,q)}(r) = \mathcal{N}\, r^{1/2-q\,\alpha}\, e^{-r^2/2}\, L_n^{(-q\,\alpha)}(r^2)\,, \tag{261}$$

where the integration path $r = x + \mathrm{i}\epsilon$ is a straight line. For $\epsilon < 0$ it lies in the lower half of the complex plane.

In the spirit of the Kustaanheimo–Steifel mapping of harmonic oscillators on Coulombic bound states we now have to define a complex variable t as a rescaled square of $r(x)$ such that the resulting path $t(x)$ remains $\mathcal{PT}$ invariant. In the $\mathcal{PT}$ symmetric setting this mapping is [P8]

$$r^2 = -2\,i\,\kappa^2 t\ , \tag{262}$$

where $\kappa = \kappa_n > 0$ depends on the individual state. This maps the straight line $r(x) = x + \mathrm{i}\epsilon$ upon the curve $t(x) = u + \mathrm{i}v$, where $u = u(x) = -x\epsilon/\kappa^2$ and $v = v(x) = (x^2 - \epsilon^2)/(2\kappa^2)$, so for $\epsilon < 0$ it forms an upwards-oriented parabola $v = -\epsilon^2/(2\kappa^2) + u^2\kappa^2/(2\epsilon^2)$ in the complex plane.

Having achieved a $\mathcal{PT}$ symmetry in the complex plane of t, we may move to the (trivial) insertions and conclude that all the above-mentioned harmonic oscillator bound-state solutions are in a one-to-one correspondence with the solutions of the Coulombic Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\alpha^2-\frac{1}{4}}{t^2}+\mathrm{i}\frac{Z\,\mathrm{e}^2}{t}\right)\Psi(t)=E\Psi(t)\ . \tag{263}$$

As in the conventional case, κ becomes $n-$ and $q-$dependent, $\kappa^2_{(n,q)}=Z\mathrm{e}^2/(2n+1-q\alpha)$, so the Coulombic solutions take the form

$$\Psi_{(n,q)}(t)=\mathcal{M}\,t^{(1-q\alpha)/2}\exp(\mathrm{i}\,\kappa^2_{(n,q)}t)L_n^{(-q\alpha)}(-2\mathrm{i}\kappa^2_{(n,q)}t)\ , \tag{264}$$

and their energy spectrum is specified by the elementary formula

$$E_{(n,q)}=\kappa^4_{(n,q)}=\frac{Z^2\mathrm{e}^4}{(2n+1-q\alpha)^2}\qquad q=\pm1,\quad n=0,1,\ldots\ . \tag{265}$$

One immediately notices the peculiarities of the $\mathcal{PT}$ symmetric Coulomb problem. First, the charge is replaced by an imaginary quantity, and consequently, the energy eigenvalues are positive, rather than negative. As for other $\mathcal{PT}$ symmetric potentials, there are now two sets of levels, and the energy of some $q=-1$ states can even become divergent for $n=2\alpha-1/2$ [P8].

This transformation can also be used to generate non-shape-invariant potentials too, similarly to the Hermitian case. In particular, the transformation discussed in subsection 3.1.3, i.e. the one taking the Eckart potential into the Natanzon type $V^{(DKV)}(x)_1$ in (120) can also be implemented in the $\mathcal{PT}$ symmetric setting [P12]. Without presenting the details, we just state the main results. To repeat the procedure in subsection 3.1.3 one again considers first the original $\mathcal{PT}$ symmetric (Eckart) potential defined on the straight $x+\mathrm{i}\epsilon$ line of the complex x plane, and then transforms this into a bent trajectory by substituting it into the function defining the variable transformation. This is now defined implicitly by $\sinh(x+\mathrm{i}\epsilon)=-\mathrm{i}\mathrm{e}^{\mathrm{i}\xi}$, leading to

$$V(\xi)=\frac{3}{4(1-\mathrm{e})^{2\mathrm{i}\xi}}+\frac{2\mathrm{i}\beta}{(1-\mathrm{e}^{2\mathrm{i}\xi})^{1/2}}-\frac{C}{1-\mathrm{e}^{2\mathrm{i}\xi}}\ , \tag{266}$$

which is the $\mathcal{PT}$ symmetric version of the $V_1^{(DKV)}(x)$ potential discussed in subsection 3.1.3. Similarly to the Hermitian case, the energy eigenvalues are again determined by a cubic formula for n, however, it turns out that in the $\mathcal{PT}$ symmetric setting there are two real roots leading to normalizable solutions [P12]. This is in accordance with the general observation concerning $\mathcal{PT}$

symmetric potentials, i.e. that the energy spectrum becomes richer (develops a second set of normalizable states), mainly due to the less strict boundary conditions.

Finally, we briefly mention another type of solvable potential which formally does not belong to the Natanzon class, but it illustrates the mechanism of spontaneous $\mathcal{PT}$ symmetry breaking. This is the $\mathcal{PT}$ symmetric square well potential defined on a finite interval, say $x \in (-1, 1)$ as

$$V(x) = \begin{cases} \mathrm{i}Y & x < 0 \\ -\mathrm{i}Y & x > 0. \end{cases} \tag{267}$$

where Y is a real constant. It is defined together with the boundary conditions $\psi(\pm 1) = 0$ [150, P14]. In [150] the case of unbroken $\mathcal{PT}$ symmetry was considered, when the energy eigenvalues are real. It was found that this holds until a critical value of Y, where the first two levels "merge and disappear". However, it can be shown that E_0 and E_1 simply become the complex adjoint of each other [P14], in accordance with the mechanism of the spontaneous breaking of $\mathcal{PT}$ symmetry.

Without presenting the details here we only sketch the main elements of the analysis in [P14]. First $\psi_0(x)$ and $\psi_1(x)$ are written in the form of hyperbolic functions as $\psi \sim a \cosh kx + b \sinh kx$, and then the wave number is matched to the (complex) energies E and Y. Then the boundary conditions are implemented through the logarithmic derivative of the solutions, and this ultimately leads to a (complex) transcendental equation

$$\lambda \coth \lambda + \kappa^* \coth \kappa^* = 0 \ , \tag{268}$$

where $\lambda^2 = -e - \mathrm{i}\varepsilon - \mathrm{i}Y$ and $\kappa^2 = -e + \mathrm{i}\varepsilon - \mathrm{i}Y$, and $E = e \pm \mathrm{i}\varepsilon$ are the complex conjugated energy eigenvalues E_0 and E_1. The transcendental equation can be solved graphically, and one finds that the energies E_1 and E_2 become complex (i.e. complex conjugates of each other) when Y reaches the dritical value $Y_{\mathrm{crit}} \sim 4.475$ [P14]. A second critical value was also found near 12.80155. This means that the energy eigenvalues turn into complex pairs at different critical coupling constants, which is different from what we have seen for the $\mathcal{PT}$ symmetric shape-invariant potentials in subsections 3.4.1 and 3.4.2. There *all* the energies turned into complex pairs at the same coupling parameters, so the spontaneous breakdown of $\mathcal{PT}$ symmetry took place in a single step, rather than continuously.

3.5 The interrelation of the three symmetry concepts

The unusual features related to the $\mathcal{PT}$ symmetry of quantum mechanical potentials naturally raise the question how other symmetries of the same potentials are affected by $\mathcal{PT}$ symmetry. We are particularly interested in constructions based on supersymmetric quantum mechanics (SUSYQM) [3, 4] and Lie algebras. The doubling of the basis states due to $q = \pm 1$ implies that the superpotential also has to carry the quasi-parity quantum number, and also that some algebras associated with the basis states of conventional potentials have to be enlarged.

$\mathcal{PT}$ symmetry and supersymmetry

Let us modify the standard SUSYQM formalism by adding the q quasi-parity quantum number to the SUSYQM shift operators

$$A^{(q)} = \frac{\mathrm{d}}{\mathrm{d}x} + W^{(q)}(x) \qquad\qquad A^{\dagger(q)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W^{(q)}(x) \tag{269}$$

through the superpotential $W^{(q)}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi^{(q)}_{0,-}(x)$, where $\psi^{(q)}_{n,-}(x)$ is the n'th normalizable wavefunction with quasi-parity q [P17, C4]. Substituting directly $A^{(q)}$ and $A^{\dagger(q)}$ in the factorized form of the Hamiltonian, the two sets of solutions would belong to two potentials shifted with respect to each other with an energy constant, because the ground-state energies $E^{(\pm q)}_{0,-}$ would be zero for $q = 1$ and -1 alike, by construction [3, 4]. In order to avoid this, let us write the "bosonic" Hamiltonian in the factorized form $\mathbf{H}_- = A^{\dagger(q)}A^{(q)} + \varepsilon^{(q)} = A^{\dagger(-q)}A^{(-q)} + \varepsilon^{(-q)}$, containing the q-dependent factorization energies $\varepsilon^{(\pm q)} = E^{(\pm q)}_{0,-}$. Then $\mathbf{H}_-$ becomes independent of q, and its eigenvalue equation takes the form

$$\mathbf{H}_-\psi^{(q)}_{n,-} = [A^{\dagger(\pm q)}A^{(\pm q)} + \varepsilon^{(\pm q)}]\psi^{(q)}_{n,-} = E^{(q)}_{n,-}\psi^{(q)}_{n,-} \ . \tag{270}$$

The "fermionic" partner Hamiltonians $\mathbf{H}^{(\pm q)}_+$, however, will depend on q:

$$\mathbf{H}^{(\pm q)}_+\psi^{(q)}_{n,+} = [A^{(\pm q)}A^{\dagger(\pm q)} + \varepsilon^{(\pm q)}]\psi^{(q)}_{n,+} = E^{(q)}_{n,+}\psi^{(q)}_{n,+} \ . \tag{271}$$

With equations (270) and (271) one can easily prove the $A^{(\pm q)}\psi^{(q)}_{n,-}$ functions are eigenfunctions of the $\mathbf{H}^{(\pm q)}_+$ "fermionic" Hamiltonians, and the corresponding energy eigenvalues are the same as those of the q-independent "bosonic" Hamiltonian:

$$\mathbf{H}^{(q)}_+A^{(q)}\psi^{(q)}_{n,-} = E^{(q)}_{n,-}A^{(q)}\psi^{(q)}_{n,-} \ , \tag{272}$$

$$\mathbf{H}^{(-q)}_+A^{(-q)}\psi^{(q)}_{n,-} = E^{(q)}_{n,-}A^{(-q)}\psi^{(q)}_{n,-} \ . \tag{273}$$

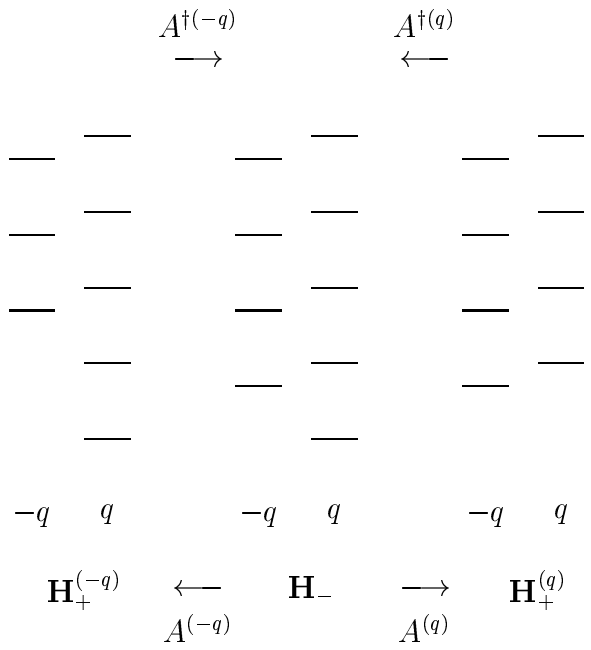


Figure 6: Schematic illustration of the relation between the spectra of the "bosonic" Hamiltonian $\mathbf{H}_-$ and its two "fermionic" partners $\mathbf{H}_+^{(q)}$ and $\mathbf{H}_+^{(-q)}$. The energy scale and the relative spacing of the energy levels is arbitrary.

However, there is a difference between (272) and (273) that in the former case $A^{(q)}\psi_{n,-}^{(q)} = 0$ holds by construction, so the partner of the ground-state "bosonic" level is missing from the spectrum of $\mathbf{H}_+^{(q)}$ [3, 4], while the situation is different for (273), so there the number of levels is the same in the "bosonic" and "fermionic" Hamiltonians. The situation is schematically illustrated on figure 6.

We illustrate this procedure with the example of the Scarf II potential (244), considering it the "bosonic" potential. Then the superpotential is [P17]

$$W^{(q)}(x) = -\frac{1}{2}(q\alpha + \beta + 1)\tanh x - \frac{\mathrm{i}}{2}(\beta - q\alpha)\mathrm{sech}x \ , \qquad (274)$$

which generates (244) as the "bosonic" potential $V_-(x)$, provided that the the factorization energies are $\varepsilon^{(q)} = -\frac{1}{4}(q\alpha + \beta + 1)^2$. The "fermionic" partner

potentials then take the form [C4]

$$V_+^{(q)}(x) = -\frac{1}{\cosh^2 x}\left[\left(\frac{q\alpha+\beta+2}{2}\right)^2+\left(\frac{q\alpha-\beta}{2}\right)^2-\frac{1}{4}\right] + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta+q\alpha+2}{2}\right)\left(\frac{\beta-q\alpha}{2}\right) \quad . \tag{275}$$

The results obtained for the Scarf II potential have significantly different implications for unbroken and broken $\mathcal{PT}$ symmetry, corresponding to real and imaginary values of α [P17, C4]. In the former case the "fermionic" partner potentials (275) are $\mathcal{PT}$ symmetric, and the energy eigenvalues remain real. In the latter case, however, the coupling parameters of both the even and odd component of the potential become complex due to the imaginary value of α, therefore the "fermionic" potentials cease to be $\mathcal{PT}$ symmetric.

We note that a similar system of partner potentials has been obtained [151] from two essentially different supersymmetric constructions; i.e. the para-supersymmetric scheme (where a three- rather than two-dimensional matrix representation is used) and second-order supersymmetry (where A and $A^\dagger$ in (269) are second- rather than first-order differential operators).

As a further combination of $\mathcal{PT}$ symmetry and supersymmetry we mention a realization of the $N = 2$ SUSYQM algebra (17) in which the supersymmetric charge and shift operators contain the time reflection (i.e., complex conjugation) operator $\mathcal{T}$ in the form [152]

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{T}A^{(q)} & 0 \end{pmatrix} \qquad \widetilde{Q}^\dagger = \begin{pmatrix} 0 & A^{\dagger(q)}\mathcal{T} \\ 0 & 0 \end{pmatrix}. \tag{276}$$

Consequently, the SUSY Hamiltonian is different in its "fermionic" component

$$\widetilde{\mathcal{H}} = \begin{pmatrix} \widetilde{H}_-^{(q)} & 0 \\ 0 & \widetilde{H}_+^{(q)} \end{pmatrix} \equiv \begin{pmatrix} A^{\dagger(q)}A^{(q)} & 0 \\ 0 & \mathcal{T}A^{(q)}A^{\dagger(q)}\mathcal{T} \end{pmatrix} . \tag{277}$$

This indicates that the "bosonic" component of the modified Hamiltonian is the same as in the original case, $\widetilde{H}_-^{(q)} = H_-^{(q)}$, while the "fermionic" component of the modified Hamiltonian coincides with the complex conjugate of the original "fermionic" Hamiltonian $\widetilde{H}_+^{(q)} = \mathcal{T}H_+^{(q)}\mathcal{T}$. By introducing the shifted energy scale as in (270) and (271) these relations become $\widetilde{\mathbf{H}}_- = \mathbf{H}_-$ and $\widetilde{\mathbf{H}}_+^{(q)} = \mathcal{T}\mathbf{H}_+^{(q)}\mathcal{T} + [\varepsilon^{(q)}]^*$. For unbroken $\mathcal{PT}$ symmetry of $\mathbf{H}_-$, i.e. when the energy eigenvalues are real and consequently $\varepsilon^{(q)}$ is also real, this means that

the energy eigenvalues of $\widetilde{\mathbf{H}}_+^{(q)}$ are also real, while for spontaneously broken $\mathcal{PT}$ symmetry, when the energy eigenvalues and $\varepsilon^{(q)}$ are complex, the energy eigenvalues of $\widetilde{\mathbf{H}}_+^{(q)}$ are the complex conjugates of the eigenvalues of $\mathbf{H}_+^{(q)}$. The eigenfunctions are equally trivially related to the original "fermionic" eigenfunctions in both cases.

Furthermore, the $\mathcal{PT}$ invariance leads to a special relation between the $\mathcal{P}$ and $\mathcal{T}$ operations themselves. If $\mathbf{H}_+^{(q)}$ is $\mathcal{PT}$ symmetric, then the complex conjugation operation has the same effect on it as the $\mathcal{P}$ spatial reflexion operation, so $\widetilde{\mathbf{H}}_+^{(q)}$ contains the spatially reflected potential appearing in $\mathbf{H}_+^{(q)}$, so the modified SUSY construction does not differ essentially from the usual one. A similar relation holds between the eigenfunctions, if they are eigenfunctions of the $\mathcal{PT}$ operator, i.e. if the $\mathcal{PT}$ symmetry is unbroken. The energy eigenvalues of $\widetilde{\mathbf{H}}_+^{(q)}$ are real and the same as those of $\mathbf{H}_+^{(q)}$, as we have seen above. In the case of spontaneously broken $\mathcal{PT}$ symmetry the situation is different since the eigenfunctions are not invariant under the $\mathcal{PT}$ operation anymore. The energy eigenvalues remain the same since the complex conjugate pairs simply transform into themselves under complex conjugation. However, in the case of the spontaneously broken $\mathcal{PT}$ symmetry, the $\mathcal{PT}$ invariance of $\widetilde{\mathbf{H}}_- = \mathbf{H}_-^{(q)}$ need not lead to the $\mathcal{PT}$ invariance of $\widetilde{\mathbf{H}}_+^{(q)}$ as we have seen on the example of the Scarf II potential [P17].

$\mathcal{PT}$ symmetry and potential algebras

Let us now turn to the algebraic framework to describe $\mathcal{PT}$ symmetric potentials. In particular, we investigate potential algebras (discussed in subsection (2.3), the ladder operators of which connect degenerate states of potentials with different depth, but of the same type. In the Hermitian case the practical equivalence of the SUSYQM construction and the one based on an su(1,1) (or su(2)) potential algebra has been demonstrated [69] for B and A class shape-invariant potentials, which contain the Morse potential and various Scarf and Pöschl–Teller potentials. We focus on the $\mathcal{PT}$ symmetric versions of type A (or PI class) potentials.

Our first results concerned the construction of an su(1,1)$\simeq$so(2,1) algebra related to the Scarf II potential (244) [P10]. It turned out that similarly to the Hermitian case, the normalizable states of this potential supply a basis for the irreducible representations of the SU(1,1) potential group. A major difference, however, is the presence of the second set of normalizable solutions due to the q quasi-parity quantum number, which indicated that a second su(1,1)$\simeq$so(2,1) algebra is required for the complete description of the problem. In the Hermitian case the second set of solutions corresponded to resonance

states with complex energy, and can be associated with finite dimensional non-unitary irreducible representations of SU(1,1) [59]. These states can be identified with the poles of the transmission amplitude (259) with $k = \pm\frac{\mathrm{i}}{2}(\alpha - \beta) - \mathrm{i}(n + \frac{1}{2})$ [P10]. In the $\mathcal{PT}$ symmetric case these states turn into bound states in the sense that their energy eigenvalues become real, unless the $\mathcal{PT}$ symmetry is broken spontaneously (i.e. α becomes imaginary rather than real). Note that in the latter situation both sets of normalizable states have complex energy eigenvalues, the energies of which are complex conjugates of each other. In group theoretical terms this means that the nature of the SU(1,1) irreducible representations also changes when one goes from the Hermitian case to the $\mathcal{PT}$ symmetric one, and also when $\mathcal{PT}$ symmetry is spontaneously broken. We note that the Scarf II potential has also been analyzed in terms of the complex sl(2,C) algebra [146, 153].

Based on the first findings we performed a systematic study of the so(2,2) $\sim$ so(2,1)$\oplus$so(2,1) algebra associated with PI type potentials, with special attention to their $\mathcal{PT}$ symmetric versions [P15]. For this we considered the differential realization of the so(2,2) algebra

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2aJ_z \ , \qquad [J_i, K_j] = 0 \tag{278}$$

$$[K_z, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2bK_z \ , \qquad i, j = +, \ -, \ z \ , \tag{279}$$

which also includes the so(4) and so(3,1) algebras for $a = b = -1$ and $a = -b = \pm 1$, respectively. We parametrized the generators as [P15]

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm h_1(x)\frac{\partial}{\partial x} \pm g_1(x) + f_1(x)J_z + c_1(x) + k_1(x)K_z\right) \ , \tag{280}$$

$$J_z = -\mathrm{i}\frac{\partial}{\partial \phi} \tag{281}$$

and

$$K_\pm = \mathrm{e}^{\pm \mathrm{i}\chi}\left(\pm h_2(x)\frac{\partial}{\partial x} \pm g_2(x) + f_2(x)J_z + c_2(x) + k_2(x)K_z\right) \ , \tag{282}$$

$$K_z = -\mathrm{i}\frac{\partial}{\partial \chi} \ . \tag{283}$$

and found that the algebra defined in (278) and (279) is obtained if the following relations hold:

$$k_2^2 - h_2 k_2' = b \qquad h_2 f_2' - f_2 k_2 = 0 \qquad k_2^2 - f_2^2 = b \ , \tag{284}$$

$$c_1 = c_2 = 0 \ , \tag{285}$$

$$h_1 = Ah_2 \qquad f_1 = Ak_2 \qquad k_1 = Af_2 \qquad g_1 = Ag_2 \ , \tag{286}$$

$$A^2 = \frac{a}{b} = \pm 1 \ . \tag{287}$$

Here we have assumed that $h_i(x) \neq 0$, $k_i \neq 0$ and $f_i \neq 0$ holds. For $h_i(x) = 0$ the differential term with respect to x would be cancelled in $J_\pm$ (280) and $K_\pm$ (282), while $f_i(x) = k_i(x) = 0$ would contradict (284). We also note that from the three equations in (284) only two are independent and that the choice of $h_2(x)$ determines $f_2(x)$ and $k_2(x)$ immediately. However $h_2(x)$ does not determine $g_2(x)$, so there are two independent functions defining this construction.

The Casimir invariant

$$C_2^{(JK)} = 2C_2^{(J)} + 2C_2^{(K)} \equiv 2\left(-aJ_+J_- + J_z^2 - J_z - bK_+K_- + K_z^2 - K_z\right) \tag{288}$$

is a second-order differential operator

$$\begin{aligned} C_2^{(JK)}\Psi \ &= \ 4bh_2^2\Psi'' + 4bh_2(h_2' + 2g_2 - k_2)\Psi' + [4b(h_2g_2' + g_2^2 - k_2g_2) \\ &\quad +2(1 - bk_2^2 - bf_2^2)(J_z^2 + K_z^2) - 8bf_2k_2J_zK_z]\Psi. \end{aligned} \tag{289}$$

The eigenfunctions of $C_2^{(JK)}$, which are also the eigenfunctions of J_z and K_z are $\Psi \equiv \Psi(x, \phi, \chi) = \mathrm{e}^{\mathrm{i}(m\phi + m'\chi)}\psi(x)$. Here $\psi(x)$ is the physical wavefunction depending on the coordinate x, while ϕ and χ are auxiliary variables, which are multiplied with m and m', the eigenvalues of generators J_z and K_z, respectively.

Since the above algebras are of rank 2, they admit a second Casimir invariant, which can be written as the difference of the two SO(2,1) Casimir invariants in (288)

$$\tilde{C}_2^{(JK)} = 2C_2^{(J)} - 2C_2^{(K)}. \tag{290}$$

It turns out that the eigenvalue of this operator is always zero for the present differential realization of the algebra, irrespective of a and b. Therefore, we have generated the symmetric irreducible representation of so(2,2) (or so(4)) [73], usually labelled as $(\omega, 0)$, where ω is the quantum number defining the eigenvalue of the first Casimir invariant

$$C_2^{(JK)}\Psi \ = \ \omega(\omega + 2)\Psi. \tag{291}$$

ω is connected with the eigenvalue $j(j+1)$ of the Casimir invariant of so(2,1) (or so(3)) by the relation $\omega = 2j$. Of course, a simple formal transition from an

so(2,1) algebra to an so(3) algebra can be made by multiplying the h_i, g_i, f_i, k_i, c_i ($i = 1, 2$) functions with the imaginary unit i. This exactly corresponds to the changes $a \to -1$ and $b \to -1$. It also turned out from the construction that only the $a = b = \pm 1$ choice leads to solvable potentials, therefore in this scheme only the so(2,2) and so(4) algebras can be obtained, but not so(3,1) [P15]. This is a constructive proof of an assumption used in [72, 73].

Following the method presented in subsection 2.3 for the su(1,1) algebra, the Schrödinger equation can be obtained from the egeinvalue equation of the Casimir invariant in case the linear derivative term is eliminated with the extra constraint

$$g_2 = \frac{1}{2}(k_2 - h_2') \ . \tag{292}$$

With this choice we get

$$\begin{aligned} C_2^{(JK)}\Psi &= 4bh_2^2\Psi'' + [b((h_2')^2 + k_2^2 - 2h_2''h_2) - 2 \\ &\quad +4(1 - bk_2^2)(J_z^2 + K_z^2) - 8bf_2k_2J_zK_z]\Psi \\ &= \omega(\omega + 2)\Psi \ . \end{aligned} \tag{293}$$

A Schrödinger-type differential equation can be obtained from (293) if h_2 is a constant. Similarly to the su(1,1) case (see [69] and subsection 2.3), this choice defines the so(2,2) algebra as a potential algebra, with generators laddering between degenerate states of potentials with different parameters but similar shape. In [P15] all members of the PI potential class (see table 1) were discussed in terms of the so(2,2) (or so(4)) potential algebra.

The ladder operators of the so(2,2) potential algebra associated Scarf II potential are

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \tanh x(J_z \pm \frac{1}{2}) + \frac{\mathrm{i}}{\cosh x}K_z\right) \ , \tag{294}$$

$$K_\pm = \mathrm{e}^{\pm \mathrm{i}\chi}\left(\pm\frac{\partial}{\partial x} - \tanh x(K_z \pm \frac{1}{2}) + \frac{\mathrm{i}}{\cosh x}J_z\right) \ . \tag{295}$$

The m and m' labels are expressed in terms of the potential parameters as $m = -(\alpha + \beta)/2$ and $m' = (\beta - \alpha)/2$. The ladder operators shift these values with one unit, which corresponds to changing α and β in a correlated way [P15].

The $\mathcal{PT}$ symmetric versions of the remaining PI class potentials were also derived in a similar way [P15], implementing also the imaginary coordinate

shift discussed in subsection 3.4.1. It was also found that the so(2,2) generators transform under the $\mathcal{PT}$ operation in a characteristic way:

$$\mathcal{PT}(J/K)_{\pm}(\mathcal{PT})^{-1} = (J/K)_{\mp} \ , \qquad \mathcal{PT}(J/K)_{z}(\mathcal{PT})^{-1} = -(J/K)_{z} \ . \quad (296)$$

As it can be seen from (294) and (295), the structure of the so(2,2) generators is essentially the same as that of the supersymmetric shift operators A and $A^{\dagger}$. In fact, direct calculation also shows that these operators have the same effect on the wavefunctions in the two symmetry-based schemes [P15, P17, C4].

It has to be noted that the so(2,2) algebra (or its compact version so(4)) plays the role of a potential algebra only for a limited number of potentials, i.e. for the members of the PI class (factorization type A), while the supersymmetric construction presented here for $\mathcal{PT}$ symmetric potentials might have wider applicability. This is somewhat different for the Morse potential belonging to the LIII class (factorization type B): it is possible to define an sl(2,C) potential algebra associated with it [146, 153], but it is not $\mathcal{PT}$ symmetric. It can be made $\mathcal{PT}$ symmetric by defining it along a bent contour of the complex x plane [91, P9].

4 Summary

The main results of the present dissertation can be summarized as follows.

1. I generalized the formalism of the factorization method by introducing spin degrees of freedom in the quantum mechanical Hamiltonian, in addition to local potential terms [P6]. I pointed out that in all three examples (of which two are new) considered, the Hamiltonians possess an infinitely degenerate ground-state energy level. I generalized the Dirac oscillator [P2], and demonstrated that it represents an example for the intimate relation of supersymmetry (factorization) and the Dirac equation.

2. I generalized the formalism of supersymmetric quantum mechanics to complex potentials and exemplified it with the complex Pöschl–Teller potential [P3] in order to aid numerical calculations generating complex phase-equivalent potentials by eliminating unphysical states [P3]. Considering supersymmetric transformations which change the spectrum by eliminating or adding bound states at specific energies or leave it unchanged, I determined closed expressions for potentials phase-equivalent with the generalized Pöschl–Teller [C2] and the more general, Natanzon-class generalized Ginocchio potential [P4].

3. I discussed two special limits of the Ginocchio potential, in which it takes the form of the hyperbolic and trigonometric version of the Pöschl–Teller potential, and pointed out that the su(1,1) algebra associated with the Ginocchio potential reduces to an su(1,1) potential algebra and an su(2) spectrum generating algebra, respectively [C3]. I pointed out for the first time that a specific unitary irreducible representation (called the supplementary series) of the SU(1,1) spectrum generating group associated with the trigonometric version of the Pöschl–Teller potential corresponds to potentials possessing a "weakly attractive" x^2-type singularity [C1], for which both independent solutions are regular near $x = 0$.

4. In a systematic study of the $\mathcal{PT}$ symmetric version of shape-invariant potentials I defined conditions for the parameters under which the normalizable states belong to energy eigenvalues that are purely real [P9], or are arranged into complex conjugated pairs [P13]. I demonstrated that except for the Morse and the Coulomb potentials, the $\mathcal{PT}$ symmetric version of the shape-invariant potantials can be obtained by an imaginary shift of the coordinate: $x \to x + \mathrm{i}\epsilon$, and also that these potentials possess richer spectrum than their Hermitian counterparts [P9]. I showed that

tuning the potential parameters of the $\mathcal{PT}$ symmetric (complex) square well, the energy eigenvalues turn from real values to complex pairs one by one, i.e. the spontaneous breaking of the $\mathcal{PT}$ symmetry occurs continuously [P14], in contrast with the case of shape-invariant potentials.

5. Analyzing the generalized Coulomb potential, which contains both the harmonic oscillator and Coulomb potential as special cases, I presented a novel approach to the Coulomb–oscillator connection in various spatial dimensions [P5]. I pointed out that the complications arising due to the singularity of the one-dimensional Coulomb problem can be avoided with the use of the generalized Coulomb potential [P5]. I introduced a generalization of the Coulomb–Sturmian basis and an su(1,1) algebra associated with it [P5]. I discussed the Coulomb–oscillator connection for the $\mathcal{PT}$ symmetric versions of these potentials too [P8]. I analyzed further "implicit" potentials with various shapes to illustrate the spectral properties, singularities [P1] and the origins [P1, P18] of Natanzon potentials.

6. I showed that a class of potentials classified previously as a conditionally exactly solvable (CES) problem is, in fact, a representative of the (exactly solvable) Natanzon potential class, with the property that in case its energy eigenvalues are real they are supplied by one of the roots of an algebraic equation cubic in the principal quantum number n [P11]. Investigating the $\mathcal{PT}$ symmetric version of this potential I showed that in case it has real energy eigenvalues, these are supplied by *two* roots, in agreement with the observation that $\mathcal{PT}$ symmetric potentials possess a richer spectrum than their Hermitian counterparts [P12]. I constructed another class of conditionally exactly solvable potentials by means of supersymmetric transformations that eliminate the ground state, add a new one or leave the spectrum (but not the potential) unchanged [P7]. I showed that the conditionally exactly solvable potentials derived this way are beyond the Natanzon potential class. Besides reproducing known results, this systematic construction also produced new potentials.

7. Investigating the $\mathcal{PT}$ symmetric version of the Scarf II potential and the su(1,1) potential algebra associated with it, I demonstrated that a second set of normalizable solution evolves from states that are resonances in the Hermitian version of the potential, and this set has another su(1,1) potential related to it [P10]. I demonstrated that this doubling of the normalizable states and algebras, which requires the introduction of the

quasi-parity quantum number $q = \pm 1$, occurs for all the shape-invariant potentials that have su(1,1) or su(2) potential algebras associated with them, and the two algebras can be unified into an so(2,2) or so(4) potential algebra [P15]. I presented for the first time analytical expressions for the the pseudo-norm of the normalizable states of the Scarf II potential, and showed that according to the expectations, it has indefinite sign [P16]. As a side result of these investigations I also determined for the first time the normalization coefficients for the bound-state wavefunctions of the Hermitian Scarf II potential [P16]. As a byproduct of these investigations, I derived a previously unknown mathematical formula for the summation of three binomial coefficients.

8. Taking the normalizable solutions of $\mathcal{PT}$ symmetric potentials associated with quasi-parity $q = +1$ and $q = -1$, I constructed a supersymmetric sheme in which the original potential has *two* supersymmetric parner potentials carrying the $q = \pm 1$ quantum numbers [P17]. I demonstrated that in case the $\mathcal{PT}$ symmetry of the original potential is spontaneously broken, its two partner potentials cease to be $\mathcal{PT}$ symmetric [P17]. I presented examples for potentials that possess a potential algebra, $\mathcal{PT}$ symmetry and supersymmetry, and discussed the interrelation of these symmetry concepts [C4].

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